# Accurate Guaranteed State Estimation for Uncertain LPVs using Constrained Convex Generators

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***Abstract*— Guaranteed state estimation for autonomous vehicles in GPS-denied areas that resort to landmarks detection and onboard sensors requires set-membership techniques that are capable of representing heterogeneous bounds using hyperplanes and ellipsoids. Recently, in the literature, the concept of Convex Constrained Generators (CCGs) has been introduced for the case where the dynamical system can be represented by a Linear Parameter-Varying (LPV) model. However, in practical applications, dynamics have uncertain parameters caused by noise-corrupted measurements of quantities of interest such as mass or orientation angles. In this paper, we first explore a closed-form solution for the convex hull of polytopes to showcase the main challenges of guaranteed state estimation for uncertain LPVs. We then propose the use of CCGs to have low conservatism when in the presence of distance measurements and avoid the exponential growth of the generators used in the state representation by performing an approximation using ray-shooting. Simulations illustrate the ability of CCGs to accurately model distance measurements with the corresponding decrease in volume without adding additional constraints.**

***Index Terms*— Uncertain Systems; Autonomous Systems; Estimation**

## I. Introduction

Missions where autonomous vehicles have onboard sensors to help their localization or use a guaranteed state estimation filter to perform collision avoidance [1], [2] can benefit from having very accurate set representations as conservative estimates would translate in very restricted movement control signals. Incorporating noise-corrupted range and bearing measurements is typically done in the literature through an over-approximation of the set resulting from range-only measurements by intervals [3] or using ellipsoids [4]. Recently, Constrained Convex Generators (CCGs) [5] allow to model this type of measurements for linear models with no uncertainties. CCGs have the advantage of allowing the representation of circular or ellipsoidal shapes as well as intersections of different convex bodies. Figure 1 showcases the intersection of a square and an ellipsoid and two ellipsoids that can be directly represented as CCGs.

The state estimation task can also be carried following the stochastic approach with Kalman filters that vary depending on the assumptions. Single beacon range measurement was tackled in [6] by a transformation of the nonlinear dynamics to obtain a Linear Time Varying (LTV) which allows for a Kalman Filter. The nonlinear model can be directly used by an Extended Kalman Filter [7]–[9]. The stochastic approach is not desirable when a guaranteed state estimation is needed as in the case of fault-tolerant control, Model Predictive approaches, or vehicle collision detection with obstacles.

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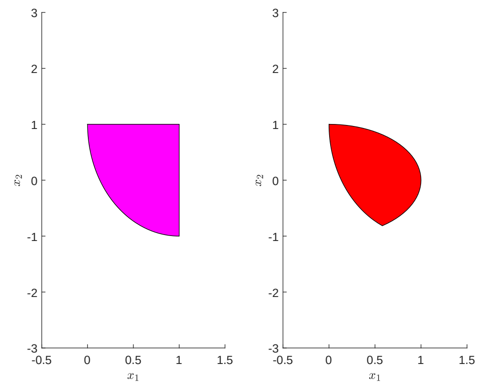


Fig. 1: Two sets that can be modeled using constrained convex generators. On the left: set resulting from the intersection of a square with an ellipse. On the right: intersection of two ellipses.

Estimation for uncertain Linear Parameter-Varying (LPV) has mostly considered polytopes such as in [10]. In the case of LTVs in discrete-time, there are proposals using intervals [11], zonotopes [12] and ellipsoids [13] which are not accurate since intersections cannot be expressed in closed-form. Techniques resorting to polytopes [14] and in the format of constrained zonotopes [15] are the relevant techniques, although uncertainties inherently point towards computing a convex hull over the trajectories for vertices of the uncertain dynamics matrix. Please note that the uncertainties can also be modeled as exogenous signals at the expenses of a larger conservatism when the uncertainty matrices do not have rank equal to the unity [16]. The case of uncertainties cannot be addressed even considering the equivalent estimation tools for nonlinear systems in [17], [18], [19], [20], [21], respectively.

The recent work in [22] has introduced various set operations using constrained zonotopes and zonotopes. Among them is the introduction of a convex hull computation in closed-form. This is a direct alternative to the employed solution using polytopes stored in the half-plane representation such as in [10]. Nevertheless, it is desirable to extend the techniques to CCGs as to encode reachable sets such as the ones in Figure 1. In this paper, we first present a closed-form expression for a CCG enclosing the union of two CCGs and then propose an order reduction method to avoid the exponential growth of the auxiliary variables within the set description. The main contributions can be highlighted as:

- Leveraging the definition for the exact convex hull for constrained zonotopes, it is shown an equivalent formulation for CCGs, which have the ability to reproduce reachable sets resulting from range and bearing measurements;
- Identifying a key caveat in the growth of auxiliary variables in the set representations, we propose to use an order reduction algorithm based on ray-shooting to decrease the computation time for the state estimation task of uncertain LPVs.

The remainder of the paper is organized as follows. Section II formalizes the state estimation problem, highlighting the exponential growth of the auxiliary variables. We review in Section III the definition and main set operations for CCGs, while Section IV is dedicated to presenting the proposed convex hull algorithm and the order reduction method. Simulations using a unicycle model for a land autonomous vehicle are provided in Section V. Conclusions and directions of future work are given in Section VI.

*Notation* : We let $0_n$ denote the $n$-dimensional vector of zeros and $I_n$ the identity matrix of size $n$. The operator $\mathrm{diag}(v)$ creates a diagonal matrix with $v$ in the diagonal or extracts the diagonal if the argument is a matrix. The transpose of a vector $v$ is denoted by $v^\intercal$, while the Euclidean norm for vector $x$ is represented as $\|x\|_2 := \sqrt{x^\intercal x}$. On the other hand, $\|x\|_\infty := \max_i |x_i|$. The cartesian product is denoted by $\times$, the Minkowski sum of two sets by $\oplus$ and the intersection after applying a matrix $R$ to the first set by $\cap_R$.

## II. Problem Statement

The problem of state estimation in uncertain LPVs can be cast as finding a set of possible values given the measurements, disturbance, noise and initial state bounds and the model is given by:

$$\begin{aligned} x(k+1) = \Big(F(\rho(k)) + \sum_{\ell=1}^{n_\Delta} \Delta_\ell(k) U_\ell\Big) x(k) + B(\rho(k))u(k) \\ + L(\rho(k))d(k) \\ y(k) = C(\rho(k))x(k) + N(\rho(k))w(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$, $d(k) \in \mathbb{R}^{n_d}$, $y(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^{n_w}$ are the system state, input, disturbance signal, output and noise, respectively. The parameter $\rho(k)$ is the part of the parameters that can be measured at time $k$, which can be treated as in the case of LTVs. The main challenge appears from considering the $n_\Delta$ uncertainties denoted by $\Delta_\ell$ and the constant matrices $U_\ell$ that account for how the uncertainties affect the nominal dynamics matrix given by $F(\rho(k))$. To lighten the notation, we will consider $F_k := F(\rho(k))$ and similarly for all the remaining matrices in (1). Notice that we have to explicitly consider $\rho$ to account for nonlinearities that enter the model in a linear fashion as will happen with unicycle model used in Section V. Moreover, in order to have a well-posed problem, we assume that all unknown signals are bounded within a compact convex set denoted by the correspondent capital letter, i.e., $x(0) \in X(0)$, $d(k) \in D(k)$ and $w(k) \in W(k)$. Without loss of generality, we will assume that $\forall k, |\Delta_\ell(k)| \leq 1$.

The problem addressed in this paper is summarized as:

*Problem 1:* Given compact convex sets $X(0)$, $D(k)$ and $W(k)$ for all $k \geq 0$ and measurements $y(k)$, how to compute a set $X(k)$ such that it is guaranteed that $x(k) \in X(k)$, $\forall k \geq 0$.

Notice that Problem 1 is called *state estimation* although a converse definition could be presented for the output of the system (this is of particular interest in sensitivity analysis [23] and system distinguishability [24]). Problem 1 is quite general in terms of the measurement set $Y(k)$, i.e., the set of all state values that conform with the measurements $y(k)$. If there is range information, $Y(k)$ is an ellipsoid; in case of bearing angles, one would get $Y(k)$ to be a convex cone; and, if we have some norm-based measurement, $Y(k)$ is an affine transformation of an $\ell_p$ unit ball.

## III. Constrained Convex Generators overview

In this section, we first review the main operations and introduce an approximation for the convex hull of the union of two CCGs. Definition 2 and Definition 3 provide a formal description of CCGs and the required operations.

*Definition 2 (Constrained Convex Generators):* A Constrained Convex Generator (CCG) $\mathcal{Z} \subset \mathbb{R}^n$ is defined by the tuple $(G, c, A, b, \mathfrak{C})$ with $G \in \mathbb{R}^{n \times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c \times n_g}$, $b \in \mathbb{R}^{n_c}$, and $\mathfrak{C} := \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{n_p}\}$ such that:

$$\mathcal{Z} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}\}.$$

*Definition 3:* Consider three Constrained Convex Generators (CCGs) as in Definition 2:

- $Z = (G_z, c_z, A_z, b_z, \mathfrak{C}_z) \subset \mathbb{R}^n$;
- $W = (G_w, c_w, A_w, b_w, \mathfrak{C}_w) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y, \mathfrak{C}_y) \subset \mathbb{R}^m$;

and a matrix $R \in \mathbb{R}^{m \times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$RZ + t = (RG_z, Rc_z + t, A_z, b_z, \mathfrak{C}_z)$$

$$Z \oplus W = \left( \begin{bmatrix} G_z & G_w \end{bmatrix}, c_z + c_w, \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_w\} \right)$$

$$Z \cap_R Y = \left( \begin{bmatrix} G_z & 0 \end{bmatrix}, c_z, \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_y\} \right).$$

Computationally speaking, it is required to store which type of generator we are using for which entries of the vector of auxiliary variables $\xi$. We would like to point out that all the aforementioned set representations are subsets of CCGs, namely:

- an interval corresponds to $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$, for a diagonal matrix $G$;

- a zonotope is given by $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$;
- an ellipsoid is defined by $(G, c, [\,], [\,], \|\xi\|_2 \leq 1)$, for a square matrix $G$;
- a constrained zonotope or polytope is $(G, c, A, b, \|\xi\|_\infty \leq 1)$;
- a convex cone in $\mathbb{R}^n$ is $(G, c, [\,], [\,], \xi \geq 0)$;
- ellipsotopes [25] are given by $(G, c, A, b, \|\xi\|_{p_1} \leq 1, \cdots, \|\xi\|_{p_m} \leq 1)$, for some $p_i > 0$, $1 \leq i \leq m$;
- AH-polytopes [26] are given by $(G, c, [\,], [\,], A\xi \leq b)$.

## IV. State Estimation for uncertain LPVs using Constrained Convex Generators (CCGs)

In this section, the state estimation strategy is presented using CCGs and introducing the necessary convex hull operation to deal with the uncertainties. The main issue arising from each of the uncertainty parameters $\Delta_\ell$ in (1) is that a product appears of the set $[-1, 1]$ with the CCG $X(k)$ when computing the set $X(k+1)$. The alternative that is typically explored is to consider the polytopic set of dynamics matrices and perform the convex hull for each of the vertices corresponding to $[-1, 1]^{n_\Delta}$ where the power of a set is understood as the cartesian product taken $n_\Delta$ times. Therefore, the propagation of the previous estimate $X(k)$ using the state equation in (1) corresponds to the set $X_{\text{prop}}(k+1)$:

$$\begin{aligned} X_{\text{prop}}(k+1) =& \operatorname{cvxHull}\left( \bigcup_{\Delta \in \operatorname{vertex}([-1,1]^{n_\Delta})} \left( F_k + \sum_{\ell=1}^{n_\Delta} \Delta_\ell(k) U_\ell \right) X(k) \right) \\ &+ B_k u(k) \oplus L_k D(k), \end{aligned} \tag{2}$$

where cvxHull computes the convex hull of the argument.

Using the measurement equation in (1) corresponds to an intersection with $Y(k+1)$ that has all possible state values that conform with $y(k+1)$, meaning an update on the estimates given as follows:

$$X(k+1) = X_{\text{prop}}(k+1) \cap_C Y(k+1).$$

### A. Convex Hull for CCGs

Let us start by defining the convex hull of two sets:

$$\begin{aligned} \operatorname{cvxHull}(Z_1, Z_2) := \big\{ z : & z = \lambda z_1 + (1-\lambda) z_2, \\ & \lambda \in [0, 1], z1 \in Z_1, z_2 \in Z_2 \big\}. \end{aligned}$$

We can now state a proposition introducing a novel over-approximation to the convex hull operation of two CCGs.

*Proposition 1:* Consider two Constrained Convex Generators (CCGs) as in Definition 2:

- $Z_1 = (G_1, c_1, A_1, b_1, \mathfrak{C}_1) \subset \mathbb{R}^n$;
- $Z_2 = (G_2, c_2, A_2, b_2, \mathfrak{C}_2) \subset \mathbb{R}^n$;

such that $\xi_1 \in \mathfrak{C}_1 \implies \alpha \xi_1 \in \mathfrak{C}_1$, for $\alpha \in [0, 1]$ and similarly for $\mathfrak{C}_2$. The CCG bounding the convex hull $Z_h = (G_h, c_h, A_h, b_h, \mathfrak{C}_h) \subset \mathbb{R}^n$ is given by:

$$G_h = \begin{bmatrix} G_1 & G_2 & \frac{c_1 - c_2}{2} & 0 \end{bmatrix}, c_h = \frac{c_1 + c_2}{2},$$

$$A_h = \left[\begin{array}{ccc|c} A_1 & 0 & -\frac{b_1}{2} & \multirow{2}{*}{0} \\ 0 & A_2 & \frac{b_2}{2} & \\ \hline I & 0 & -\frac{1}{2}\mathbf{1} & \multirow{4}{*}{I} \\ -I & 0 & -\frac{1}{2}\mathbf{1} & \\ 0 & I & \frac{1}{2}\mathbf{1} & \\ 0 & -I & \frac{1}{2}\mathbf{1} & \end{array}\right], b_h = \begin{bmatrix} \frac{1}{2} b_1 \\ \frac{1}{2} b_2 \\ -\frac{1}{2}\mathbf{1} \end{bmatrix}$$

$$\mathfrak{C}_h = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathcal{B}_\infty^{2n_{g1} + 2n_{g2} + 1}\},$$

where $\mathcal{B}_\infty^r$ is the $\ell_\infty$ unit ball for a $\xi$ auxiliary variable of size $r$.

*Proof:* We start by introducing an auxiliary variable $\xi_0 = 2\lambda - 1$ that is zero-centered to replace the parameter $\lambda$ in the convex hull definition. Let us label $\xi_1'$ and $\xi_2'$ as the auxiliary variables in $Z_1$ and $Z_2$ definitions, respectively. A point $z$ in the convex hull has to satisfy the following equality with the transformed variables $\xi_0$ and $\xi_1 = \lambda \xi_1'$ and $\xi_2 = (1-\lambda)\xi_2'$:

$$z = \frac{c_1}{2}(1 + \xi_0) + G_1 \xi_1 + \frac{c_2}{2}(1 - \xi_0) + G_2 \xi_2. \tag{3}$$

The original constraints can also be converted to involve the transformed variables $\xi_1$ and $\xi_2$ obtaining:

$$\begin{aligned} A_1 \xi_1' = b_1 \iff & A_1 \xi_1 = b_1 \frac{(1 + \xi_0)}{2} \\ \iff & A_1 \xi_1 - \frac{b_1 \xi_0}{2} = \frac{b_1}{2} \end{aligned} \tag{4}$$

and in a similar fashion for $\xi_2'$ constraints:

$$\begin{aligned} A_2 \xi_2' = b_2 \iff & A_2 \xi_2 = b_2 \frac{(1 - \xi_0)}{2} \\ \iff & A_2 \xi_2 + \frac{b_2 \xi_0}{2} = \frac{b_2}{2}. \end{aligned} \tag{5}$$

Up to this point, all transformations have been exact and equations (3), (4) and (5) written in matrix form result in:

$$\left( z = \begin{bmatrix} G_1 & G_2 & \frac{c_1 - c_2}{2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_0 \end{bmatrix} + \frac{c_1 + c_2}{2} : \right.$$
$$\left. \begin{bmatrix} A_1 & 0 & -\frac{b_1}{2} \\ 0 & A_2 & \frac{b_2}{2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} b_1 \\ \frac{1}{2} b_2 \end{bmatrix}, \{\mathfrak{C}_1, \mathfrak{C}_2, |\xi_0| \leq 1\} \right).$$

The above definition is just a part of what is presented in the statement since we are still missing constraints to enforce the relationship that $\xi_1$ and $\xi_2$ are related through $\xi_0$ in a convex sum. To that end, we need to add inequalities to constrain $\xi_1 \to 0$ when $1 - \lambda \to 1$ and be $\xi_1$ when $1 - \lambda \to 0$. These results in the inequalities:

$$\begin{aligned} -1 \leq \xi_1 + (1 - \lambda) \leq 1 \\ -1 \leq -\xi_1 + (1 - \lambda) \leq 1 \end{aligned}$$

and in a similar fashion to $\xi_2$ but using $\lambda$, resulting in:

$$\begin{aligned} -1 \leq \xi_2 + \lambda \leq 1 \\ -1 \leq -\xi_2 + \lambda \leq 1. \end{aligned}$$

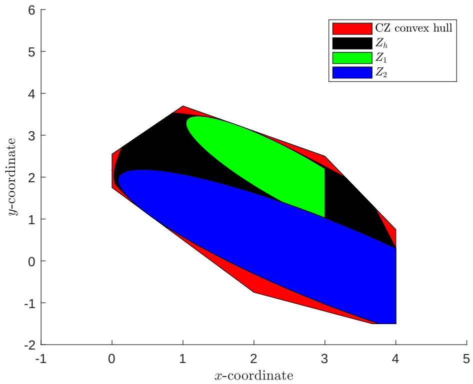


Fig. 2: Comparison between the set $Z_h$ and the convex hull that one would obtain if first converted both $Z_1$ and $Z_2$ to constrained zonotopes by overbounding all convex generators by the $\ell_\infty$ unit ball.

The above inequalities can be converted to equality constraints resorting to the use of additional auxiliary variables to serve as residuals. Rearranging the terms to have all auxiliary variables in the left-hand side and the numerical values on the right-hand side and placing them in matrix form results on the last block of $A_h$ in the statement. ■

Proposition 1 is not the exact convex hull since the last inequalities added were relaxed with the use of residual variables for a general convex generator. Figure 2 depicts an example of sets $Z_1$ and $Z_2$ with the respective set $Z_h$ as given by Proposition 1 and what one would get if first converted the sets to constrained zonotopes and then applied the exact convex hull given in [22]. As observed, even though the proposed method in Proposition 1 is not exact for CCGs, it still offers a better accuracy than computing the exact convex hull of the polytopic over-approximation of the sets.

The convex hull operator increases linearly the number of auxiliary variables by an additional $2(n_{g1} + n_{g2}) + 1$, however, this step has to be performed for all vertices which are exponential in the number of uncertainties. Such an issue was already present in [10] for polytopic set descriptions using the optimal convex hull formulation. In the following lemma, we establish an equivalent exponential growth of auxiliary variables for CCGs, even for 1 uncertainty parameter, thus pointing out a critical limitation of guaranteed state estimation for uncertain LPV systems.

*Lemma 1:* Consider a system as in (1) with a state space of dimension $n$ and a single uncertainty parameters $\Delta_1$. If the initial set $X(0)$ and the disturbance sets are represented respectively by $n_x$ and $n_d$ auxiliary variables, then, even without the intersection with the measurements, $X(k)$ requires $6^k n_x + (1+n_d)\frac{6^k-1}{5}$ auxiliary variables.

*Proof:* We start by pointing out that the linear map adds no variables, the Minkowski sum returns a set with the number of variables of both sets combined and that the convex hull returns a set with 3 times the number of variables of both sets plus $\xi_0$. The propagate phase in (2) requires computing the convex hull of two sets (one uncertainty results in 2 vertices) followed by the Minkowski sum with $D(k)$. Since the linear maps add no additional variables, we can write the recursive evolution of the number of auxiliary variables $\delta(k)$ at time $k$ as:

$$\delta(k+1) = 6\delta(k) + 1 + n_d$$

by noting that both sets within the convex hull have always the same number of auxiliary variables. Therefore, we get:

$$\begin{aligned}\delta(k) &= 6^k\delta(0) + (1+n_d)\sum_{\tau=0}^{k-1} 6^\tau \\ &= 6^k n_x + (1+n_d)\frac{1-6^k}{1-6} \\ &= 6^k n_x + (1+n_d)\frac{6^k-1}{5},\end{aligned}$$

which concludes the proof. ■

In order to keep the computation time for each iteration bounded, we introduce the order reduction in Algorithm 1, which computes a CCG with a specified number of constraints $\gamma$ using $n+\gamma$ generators which is of the form of a polytope. The procedure starts by constructing a collections hyperplanes tangent to the surface in order to have a bounding polytope $v^\intercal x \leq b$, which is then converted to the CCG representation. We remark that if the CCG is representing a polytope (i.e., it is equivalent to a CZ) and vectors in $v$ are all orthogonal to the facets of the polytope, then $X_{\mathrm{red}}(k) = X(k)$ but with a decreased order in the representation. This is a trivial observation from the fact that $v'x \leq b$ would be the exact polytope. The $\min$ and $\max$ operations are element-wise.

**Algorithm 1** Order Reduction using points on the surface.

**Require:** Set $X(K) \subseteq \mathbb{R}^n$ and desired order $\gamma$.
**Ensure:** Calculation of $X(k) \subseteq X_{\mathrm{red}}(k) \subseteq \mathbb{R}^n$ with $n_g = \gamma + n$ generators and $n_c = \gamma$ constraints.

/* *Get points* $p_i$ *on the surface such that* $p_i = \arg\max v_i^\intercal p_i,\ 1 \leq i \leq \gamma$ */
$[v, p] = \mathrm{sampleSurface}(X(k), \gamma)$
/* *Compute box* $\tilde{Z}$ *for the points* $p$ */
$\tilde{Z} = (\frac{1}{2}\mathrm{diag}(\max p - \min p), \frac{1}{2}(\max p + \min p), [\,], [\,], \|\tilde{\xi}\|_\infty \leq 1)$
/* *Calculate* $b$ *and* $\sigma$ *such that all entries* $v_i^\intercal p_i \in [\sigma, b]$ */
$\sigma = \min v^\intercal p$
$b = \mathrm{diag}(v^\intercal p)$
$X_{\mathrm{red}}(k) = ( \begin{bmatrix}\tilde{Z}.G & 0_{n\times\gamma}\end{bmatrix}, \tilde{Z}.c, \begin{bmatrix}v^\intercal \tilde{Z}.G & \frac{1}{2}\mathrm{diag}(\sigma - b)\end{bmatrix}, \frac{b+\sigma}{2} - v^\intercal \tilde{Z}.c, \|\tilde{\xi}\|_{\mathrm{inf}} \leq 1)$

## V. Simulations

In this section, simulations results are presented for a unicycle model of an autonomous vehicle in discrete-time for which there is a digital compass as an on-board sensor providing measurements of the orientation angle with a $\pm 5^\circ$ error. Simulations were run in Matlab R2018a running on a HP machine with a Intel Core i7-8550U CPU @ 1.80GHz and 12 GB of memory resorting

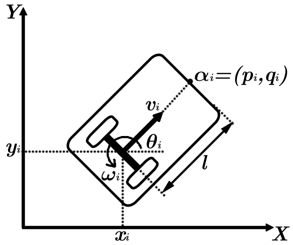


Fig. 3: Schematic of the unicycle model for the vehicles.

to Yalmip as the language to model optimization problems and Mosek as the underlying solver. Videos, figures and code can be found in `https://github.com/danielmsilvestre/CCGuncertainLPV`

We recover the example considering unicycle dynamics described in [27]. The vehicle schematic representation is given in Figure 3 and has the following dynamics in discrete-time:

$$\begin{bmatrix} p_i \\ q_i \end{bmatrix}(k+1) = \begin{bmatrix} p_i \\ q_i \end{bmatrix}(k) + \mathrm{Ts}\ A_i(\theta_i)\begin{bmatrix} v_i \\ w_i \end{bmatrix}(k)$$

where the state $(p_i, q_i)$ identify the position of the front of the $i$th vehicle and the inputs $(v_i, w_i)$ account for the linear velocity and rotation. Moreover, $\mathrm{Ts} = 0.1$ stands for the sampling time, $\theta_i$ (we omit the time dependence in $k$ for a more compact presentation) for the orientation and matrix $A_i(\theta_i)$ is given as:

$$A_i(\theta_i) = \begin{bmatrix} \cos\theta_i & -l\sin\theta_i \\ \sin\theta_i & l\cos\theta_i \end{bmatrix}.$$

In this simulation, we consider a single vehicle running for a total of 15 seconds and, assuming that the compass takes measurements $\hat{\theta}_1$ of the true variable $\theta_1$ that have a maximum of $\pm 5°$ following a uniform distribution. Therefore, at each iteration time $k$, matrix $A_1$ in the dynamics is not available to the observer and we have to consider $\hat{\theta}_1$ to generate the nominal dynamics and an uncertainty $\Delta_1$ with maximum magnitude of $5°$, which fits (1).

The trajectory-tracking control law used is:

$$\begin{bmatrix} v_i(k) \\ w_i(k) \end{bmatrix} = \frac{A_i^{-1}(\theta_i)}{\mathrm{Ts}}\left(\tau(k+1) - \frac{\tau(k)}{2} - 0.5\begin{bmatrix} p_i(k) \\ q_i(k) \end{bmatrix} + d(k)\right)$$

where $\tau(k)$ accounts for the discrete sequence of waypoints in the trajectory. Once again, we assume that there is a telemetry sensor that produces estimates corrupted by noise of the value of $p_1(k)$ and $q_1(k)$ and add the corresponding disturbance term $d(k)$ to account for those differences. Moreover, there are two beacons at positions $\begin{bmatrix} 5 & 25 \end{bmatrix}^\intercal$ and $\begin{bmatrix} 23 & 10 \end{bmatrix}^\intercal$ that can be detected within a 5 and 2 units of distance which allows to better localize the vehicle.

The vehicle performs a figure 8 trajectory such that it can only get measurements from each beacon in one time interval. Figure 4 illustrates the volume evolution for the set-valued estimates $X(k)$ when using constrained zonotopes [15] and CCGs when both used the same order reduction method in Section IV. Since the vehicle is moving and most of the time performing dead reckoning with the uncertain LPV model, the volume keeps increasing and is lowered when the vehicle reaches the beacon areas. The main trend to observe is that the added accuracy of the $\ell_2$ ball representing the range measurement from the beacon contributes to a better performance of the CCG filter.

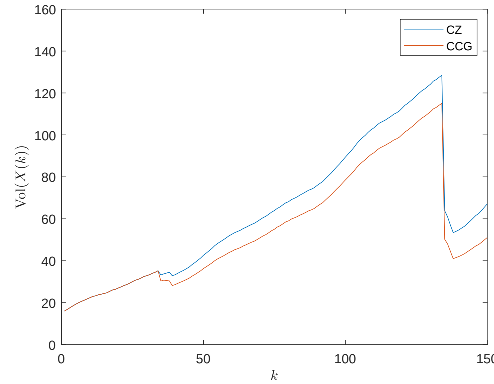


Fig. 4: Comparison of the volume for both set-valued estimates when using constrained zonotopes (CZ) and CCGs for the figure 8 trajectory.

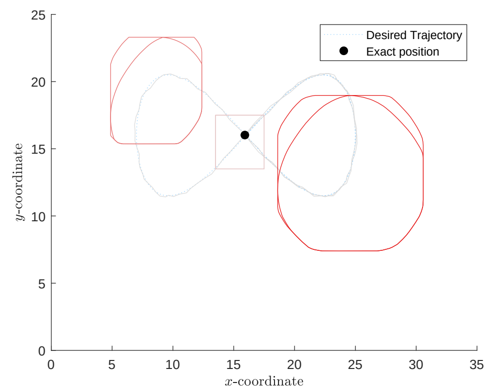


Fig. 5: Trajectory executed by the vehicle and the correspondent set-valued estimates at multiples of 65 iterations when using constrained zonotopes (CZ) and CCGs for the figure 8 trajectory.

In Figure 5, it is illustrated the trajectory executed by the vehicle and the corresponding set-valued estimates using both the CZ and CCG approaches. We have selected a small number of time instants to display the sets as to avoid cluttering the image, but the full video can be found in the GitHub repository associated with the paper.

A last relevant issue is the elapsed time in each iteration taken by both filters with different set representations. Figure 6 shows the computation times across iterations during the whole simulation. At the beginning, both filters have very similar behavior pointing out to the fact that the CCG is yet to have round facets and the order reduction produces equivalent representations. However, as the simulation progresses the set is intersected with the range measurements. The curved boundaries of the CCGs result in a more complex representation. When the vehicle finds the second beacon

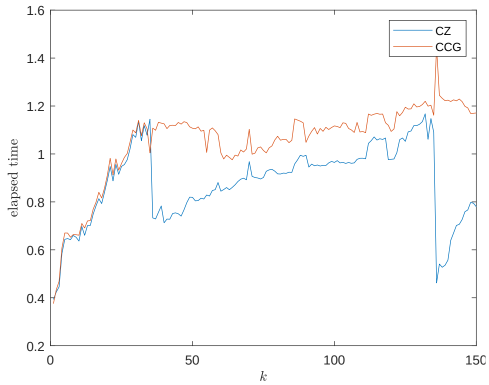


Fig. 6: Elapsed time for each iteration of both methods taking into account the constructiong of the set, approximation algorithm and volume computation.

and the set is considerably reduced in size, the CZ approach has a better performance given that $X(k)$ has a shape close to an interval, where its accuracy is the worst. This result points out to the need to further develop order reduction methods for CCGs that can exploit the nature of the sets. This is not a trivial task given the requirement of computing an outer-approximation to maintain the guaranteed feature in the estimation using set-membership approaches.

## VI. Conclusions and Future Work

In this paper, we have address the problem of set-valued estimation of autonomous vehicles with uncertainties in the dynamics. A direct example is the case of land robots that can be cast as uncertain Linear Parameter-Varying (LPV) models where the orientation angle being uncertain causes issues for techniques developed for LPVs. We then introduce a closed-form expression for convex hulls of Convex Generators (CCGs) and an order reduction algorithm.

In a simulation representing a vehicle performing dead reckoning with occasional access to range measurements from beacons, it is shown that the current proposal significantly improves the estimation quality with a smaller increase in the computational time caused mainly by the order reduction algorithm. As future work, increasing the performance of order reduction methods for CCGs that take into account the round nature of some of its facets can greatly improve the performance of the filter.

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