Distributed Observers for LTV Systems: a Distributed Constructibility Gramian Based Approach

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Abstract

The paper addresses the problem of designing distributed observers for discrete linear time-varying (LTV) systems with distributed sensor nodes. It is shown, under the conditions of collective observability, strong connectivity of the sensor communication network, and invertibility of the state transition matrix, that the resulting observer yields exponential stability of the estimation errors, with a pre-defined convergence rate, and in certain situations requires less data exchange. It is shown that for linear time-invariant (LTI) systems this method yields fixed gains that can be computed in a few steps of a distributed algorithm with the underlying communication network. A design example is given where the asymptotic performance of the proposed observer is shown to be similar to that obtained using a distributed Kalman filtering approach.

Key words: Sensor networks; Time-varying systems.

1 Introduction

1.1 Motivation

Spawned by recent advances in wireless sensor networks and distributed sensing, there has been a flurry of activity on the topic of distributed state estimation, see for example [14] and the references therein. Distributed state estimation and control have a wide range of applications, from network localization to environmental monitoring and formation control of vehicles (see [1, 6] for an introduction to these topics).

One of the most studied families of distributed estimation algorithms in discrete-time is distributed Kalman filters, which extend the theory of Kalman filtering to a distributed setting [7,8,11,15]. The concept of distributed Kalman filtering is also suitable to the problem of distributed state estimation of LTV systems. This problem is often addressed through the method of information diffusion [4, 16, 17], or with covariance intersection [3]. These methods in general require that the estimation error covariances be computed locally and exchanged among nodes. The issue of bandwidth efficiency is of paramount importance in practical applications since lower bandwidth translates into lower energy consumption and therefore increased operational autonomy. Moreover, it is difficult to obtain in-hand convergence rates for the estimation errors.

Borrowing from the theory in [3], in this paper we present an alternative design for a distributed observer for LTV systems with guaranteed stability, which requires only collective observability. However, in contrast with the method in [3], each node is required to transmit at each time a special version of the global output matrix, instead of the local information matrix. Therefore, the resulting observer requires the transmission of fewer data if the total dimension of the measurements in the network is smaller than the dimension of the system state. In addition, with this method, the convergence rate of the estimation errors is defined beforehand.

The problem of designing LTI distributed observers with fixed gains that guarantee the stability of the estimation error for general collectively observable LTI discrete-time systems has only recently been addressed [10, 12]. Most methods require in general centralized computations of the observer parameters, however, in practical applications centralized computations may not be possible if the sensors are deployed without information about the sensor network topology and global observation model. To address this problem, the works in [5, 13] allow for a distributed computation of the observer parameters in finite time methods. We show that for LTI systems the LTV observer proposed in this paper is equivalent to that in [13].

The main result of this paper is a distributed observer for

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LTV systems with guaranteed stability with the following characteristics:

- has a pre-defined convergence rate.
- requires the transmission of fewer data than the methods in [3,4] if the total dimension of the measurements in the network is smaller than the dimension of the system state.
- is equivalent to the method in [13] for LTI systems.

1.2 Paper structure

The paper is structured as follows. Section 2 formulates the problem of distributed observer design and describes the assumptions required. Section 3 describes the new estimation algorithm proposed and the distributed parameter computation algorithm for LTI systems. Section 4 contains the main theorem of this paper. To illustrate the performance of the algorithm proposed, Section 5 shows the results of the application of the estimation algorithms to an illustrative design example. Finally, Section 6 contains the conclusions of the paper.

1.3 Notation

Throughout this paper, we will use the symbol $\otimes$ for the Kronecker product. The notation $|\cdot|$ represents the cardinality of a set. The notation $\lfloor\cdot\rfloor$ represents the floor operator, or the rounding down to the closest lower integer, while $\rho(\cdot)$ denotes the spectral radius of a square matrix. I_M denotes an $M\times M$ identity matrix, and $\mathbf{1}$ represents an $N\times 1$ vector with ones in every entry. When clear from the context, the superscript of a variable, e.g. x^i, refers to the node index of that variable, where $i \in \{1,\dots,N\} := \mathcal{N}$. The operator $\mathrm{row}(\cdot)$ is defined by $\mathrm{row}(X^i) := [X^1,\dots,X^N]$, the operator $\mathrm{col}(\cdot)$ represents the column operator, i.e. $\mathrm{col}(X^i) := \mathrm{row}({X^i}^T)^T$, and the operator $\mathrm{diag}(X^i)$ yields a block diagonal matrix whose diagonal elements are $X^1,\dots,X^N$. The operator $\mathrm{trim}_n(\cdot)$ yields a matrix containing only the first n rows of the argument.

2 Problem definition

Consider the discrete autonomous dynamical system

$$x_{t+1} = A_t x_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector at time step t, and $A_t \in \mathbb{R}^{n\times n}$ is the dynamics matrix and satisfies the following assumption

Assumption A1 *The matrix A_t is invertible for all $t \geq 0$.*

We also define the following state transition matrix for any times t and t_0 such that $t > t_0$,

$$\Phi(t,t_0) := A_{t-1}\dots A_{t_0}, \tag{2}$$

$$\Phi(t_0,t) := A_{t_0}^{-1}\dots A_{t-1}^{-1}, \tag{3}$$

and therefore

$$x_t = \Phi(t,t_0)x_{t_0}. \tag{4}$$

The state vector is observed by a set of sensor nodes $\mathcal{N}$, with cardinality $N = |\mathcal{N}|$. The measurement equation associated with the generic node $i \in \mathcal{N}$ is defined as

$$y_t^i = C_t^i x_t, \tag{5}$$

where $y_t^i \in \mathbb{R}^{m_i}$ denotes the observation vector at time t, and $C_t^i \in \mathbb{R}^{l_i\times n}$ is the observation matrix. The overall network can be described by the pair $(\mathcal{N},\mathcal{A})$ where $\mathcal{A} \subseteq \mathcal{N}\times\mathcal{N}$ is the set of node pairs that denote the directed connections between the nodes. We let $\mathcal{N}^i$ be the set of in-neighbors of i, i.e., $\mathcal{N}^i := \{j : (j,i) \in \mathcal{A}\}$.

The following assumption describes the communication limitations among sensors.

Assumption A2 *At each time step, the nodes are allowed to communicate once according to the network structure defined by $\mathcal{A}$, i.e. a node i is allowed to communicate once with node j if and only if $(i,j) \in \mathcal{A}$.*

In this paper, we consider a matrix Π, henceforth referred to as the consensus matrix, whose value in row i and column j is defined as $\pi^{i,j}$, where $\pi^{i,j} = 0$ if $(i,j) \notin \mathcal{A}$. The above matrix is assumed to satisfy the following standard assumption:

Assumption A3 *The consensus matrix Π is stochastic and primitive, that is $\Pi\mathbf{1} = \mathbf{1}$, Π is nonnegative and there exists a positive integer k such that all elements of Π^k are strictly positive.*

For strongly connected graphs, a matrix Π satisfying Assumption A3 can be generated simply by taking $\pi^{i,j} = 1/|\mathcal{N}^i|$ for $j \in \mathcal{N}^i$.

To describe the necessary observability assumption we first need to define the following matrices which contain information about the available data at node i at time t from a previous time τ

$$\mathcal{C}_{t,\tau}^i := \sum_{j\in\mathcal{N}} \pi_\tau^{i,j} \bar{C}_{t-\tau-1}^{jT} \bar{C}_{t-\tau-1}^j \tag{6}$$

$$\bar{C}_t^i := \begin{cases} C_t^i & \text{if } t \geq 0 \\ \mathbf{0} & \text{if } t < 0 \end{cases}, \tag{7}$$

$$\mathcal{G}_{t,\tau}^i := \Phi(t-\tau-1,t)^T \mathcal{C}_{t,\tau}^i \Phi(t-\tau-1,t), \tag{8}$$

where $\pi_\tau^{i,j}$ is the value of Π^τ in row i and column j. The required observability assumption is the following

Assumption A4 *There exists a positive constant α and an information horizon $k \geq 0$ such that the distributed constructibility gramian*

$$\mathcal{G}_t^i := \sum_{\tau=0}^{k} \mathcal{G}_{t,\tau}^i. \tag{9}$$

satisfies $\mathcal{G}_t^i \succ \alpha I_n$ for all time $t > k$ and all nodes $i \in \mathcal{N}$

The problem of distributed state estimation that will be addressed in this paper is defined as follows.

Problem 1 Under assumptions A1-A4 and given that each node $i \in \mathcal{N}$ has access at each time t to the dynamics matrix A_t, the local measurement y_t^i, the local observation matrix C_t^i and information transmitted by the in-neighbours of i, the problem of distributed state estimation addressed in this paper consists of designing a distributed LTV observer, with parameters computed based on the global consensus matrix Π and a pre-selected global decrease rate β, such that each node reconstructs locally the state of the global system (1), as $\hat{x}_t^i$, with the estimation error $x_t - \hat{x}_t^i$ converging to zero exponentially with rate β, that is, $\|x_t - \hat{x}_t^i\| \leq a\beta^t \max_{j\in\mathcal{N}} \|x_0 - \hat{x}_0^j\|$ for some $a > 0$.

3 Algorithm

3.1 State estimate

The algorithm proposed in this paper has the following form

$$\hat{x}_{t+1}^i = A_t \left(\Omega_t^i\right)^{-1} \left(\sum_{j\in\mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j \hat{x}_t^j + C_t^{iT} y_t^i \right), \tag{10}$$

where
$$\Omega_t^i := C_t^{iT} C_t^i + \sum_{j\in\mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j \hat{x}_t^j, \tag{11}$$

and $\bar{\Omega}_t^j$ is an information matrix to be determined.

This observer algorithm is similar to the distributed Kalman filter with consensus on information given in [3]. However, in the present, we consider that the matrices $\bar{\Omega}^i$ are based on the constructibility gramian. Given a desired decrease rate β with $0 < \beta < 1$, and the information horizon $k \geq 0$ the information matrix of node i, which can be interpreted as a discounted constructibility gramian, is given by

$$\bar{\Omega}_t^i := \sum_{\tau=0}^{k} \beta^{\tau+1} \mathcal{G}_{t,\tau}^i + \chi_t, \tag{12}$$

where $\chi_t := \alpha\beta^{t+1}\Phi(0,t)^T\Phi(0,t)$. In the next subsection, we show how to compute the matrices $\bar{\Omega}_t^i$ in a distributed fashion, given that at each time step the nodes can just communicate once with their neighbours a limited amount of information.

3.2 Information matrix computation

The information matrix $\bar{\Omega}_t^i$ can be computed in a distributed fashion by noting that

$$\bar{\Omega}_t^i = \mathcal{O}_t^{iT} \Delta^i \mathcal{O}_t^i + \chi_t, \tag{13}$$

where

$$\Delta^i := \operatorname{diag}\left(\Lambda_0^i, \ldots, \Lambda_k^i\right) \tag{14}$$
$$\Lambda_\tau^i := \operatorname{diag}\left(\beta^{\tau+1}\pi_\tau^{i,1} I_{l_1}, \ldots, \beta^{\tau+1}\pi_\tau^{i,N} I_{l_N}\right). \tag{15}$$

and $\mathcal{O}_t^i$ is a constructibility matrix defined as

$$\mathcal{O}_t^i := \left[\mathcal{O}_{t,0}^{iT}, \ldots, \mathcal{O}_{t,k}^{iT}\right]^T, \tag{16}$$
$$\mathcal{O}_{t,\tau}^i := \begin{bmatrix} \mathcal{O}_{t,\tau}^{i1} \\ \vdots \\ \mathcal{O}_{t,\tau}^{iN} \end{bmatrix}, \tag{17}$$
$$\mathcal{O}_{t,\tau}^{ij} := \begin{cases} \bar{C}_{t-\tau-1}^j \Phi(t-\tau-1, t) & \text{if } \pi_\tau^{i,j} \neq 0 \\ \mathbf{0}_{l_j \times n} & \text{otherwise} \end{cases}. \tag{18}$$

It can be seen that Δ^i can be pre-computed offline with knowledge of Π and β. Therefore, to compute $\bar{\Omega}_t^i$ one needs to compute $\mathcal{O}_t^i$ at each node in a recursive and distributed fashion.

3.3 Distributed computation of $\mathcal{O}_t^i$

For each $i, j \in \mathcal{N}$ define an integer $\tau^{ij} := \min\{\tau : \pi_\tau^{i,j} \neq 0\}$, which represents the minimum time for information from node j to reach node i. Also, define an index n_{ij} such that $n_{ii} = i$ for all $i \in \mathcal{N}$ and, if $i \neq j$, $n_{ij} \in \mathcal{N}^i$ and $\tau^{ij} = \tau^{n_{ij}j} + 1$. That is, n_{ij} indicates one of the nodes from which node i receives information from node j first.

Then, with τ^{ij} and n_{ij} one can compute matrices D^{ij} and E^i given by

$$D^{ij} := \begin{bmatrix} D_1^{ij} & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & D_N^{ij} \end{bmatrix}, \tag{19}$$
$$D_p^{ij} := \begin{cases} I_{l_p} & \text{if } n_{ip} = j \\ \mathbf{0}_{l_p} & \text{otherwise} \end{cases}, \tag{20}$$

and

$$E^i := \operatorname{col}(E_0^i, \ldots, E_k^i), \tag{21}$$
$$E_q^i := \operatorname{diag}(E_{1,q}^i, \ldots, E_{N,q}^i), \tag{22}$$
$$E_{p,q}^i := \begin{cases} I_{l_p} & \text{if } q = \tau^{ip} \\ \mathbf{0}_{l_p} & \text{otherwise} \end{cases}. \tag{23}$$

Having computed D^{ij} and E^i, the distributed computation of $\mathcal{O}_t^i$ achieved by defining

$$\mathcal{T}_t^i := \begin{bmatrix} \mathcal{O}_{t,\tau^{i1}}^{i1} \\ \vdots \\ \mathcal{O}_{t,\tau^{iN}}^{iN} \end{bmatrix}, \tag{24}$$

which is computed recursively as follows

$$\mathcal{T}_{t+1}^i = \sum_{j \in \mathcal{N}^i \setminus \{i\}} D^{ij} \mathcal{T}_t^j A_t^{-1} + D^{ii} \mathcal{O}_{t+1,0}^i, \tag{25}$$

where $\mathcal{T}_{-1}^i = \mathbf{0}_{l \times n}$ for all $i \in \mathcal{N}$ and $\mathcal{O}_{t+1,0}^i$ can be computed by agent i at time t from (7), (17) and (18). Having obtained $\mathcal{T}_t^i$ one can compute $\mathcal{O}_t^i$ as

$$\mathcal{O}_{t+1}^i := \operatorname{trim}_{l(k+1)} \left(\begin{bmatrix} \mathbf{0}_{l \times n} \\ \mathcal{O}_t^i \end{bmatrix} A_t^{-1} \right) + E^i \mathcal{T}_{t+1}^i \tag{26}$$

where $l := \sum_{i \in \mathcal{N}} l_i$ and $\mathcal{O}_{-1}^i = \mathbf{0}_{l(k+1) \times n}$.

3.4 Overview

In summary, the observer can be described by the following algorithm

Algorithm 1 Observer algorithm for node i.

Input: β, α, k, $\pi^{i,j}$, Δ^j, D^{ij}, and E^j, $j \in \mathcal{N}^i$
- $\mathcal{T}_{-1}^i = \mathbf{0}_{l \times n}$
- $\mathcal{O}_{-1}^i = I_n$
- $\chi_{-1} = \alpha I_n$
- **for all** $t \geq 0$ **do**
  - $\chi_t = \beta A_{t-1}^{-1T} \chi_{t-1} A_{t-1}^{-1T}$
  - For all $j \in \mathcal{N}^i$ Obtain $\mathcal{O}_t^j$ from $\mathcal{T}_t^j$ using (26);
  - $\bar{\Omega}_t^j = \mathcal{O}_t^{jT} \Delta_t^j \mathcal{O}_t^j + \chi_t$;
  - $\Omega_t^i = C_t^{iT} C_t^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j$;
  - Send $\mathcal{T}_{t+1}^i, \hat{x}_{t+1}^i$ to out-neighbours;
  - Receive $\mathcal{T}_t^j, \hat{x}_t^j$, $j \in \mathcal{N}^i$ from in-neighbours;
  - $\mathcal{T}_{t+1}^i = \sum_{j \in \mathcal{N}^i} D^{ij} \mathcal{T}_t^j A_t^{-1} + D^{ii} \mathcal{O}_{t+1,0}^i$;
  - $\hat{x}_{t+1}^i = A_t \left(\Omega_t^i\right)^{-1} \left(\sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j \hat{x}_t^j + C_t^{iT} y_t^i \right)$;
- **end for**

In contrast with the algorithm in [3] where at each time each node transmits the state estimate $\hat{x}_t^i$ and the information matrix $\bar{\Omega}_t^i$ which is of size $n \times n$, in the algorithm proposed in this paper each node communicates its own estimate $\hat{x}_t^i$ and $\mathcal{T}_t^i$, which is of size $l \times n$, to other nodes. Therefore, this method requires the communication of less data if $n > l$.

3.5 Distributed computation of gains for LTI systems

For linear time-invariant systems, where $A_t := A$ and $C_t := C$ for all $t \geq 0$, the information matrices Ω_t^i and $\bar{\Omega}_t^i$ as proposed in this paper, can be time-invariant, that is $\Omega_t^i = \Omega^i$ and $\bar{\Omega}_t^i = \bar{\Omega}^i$ for all $t \geq 0$. As is shown in [13], the design process can be done beforehand in a distributed fashion in a finite number of steps $\bar{k} = \tilde{k} + n$, where $\tilde{k}$ is the primitivity index of Π, which is the smallest integer such that $\Pi^{\tilde{k}}$ has only strictly positive entries, and n is the dimension of the state. The process is described in the following algorithm.

Algorithm 2 Design algorithm.

Input: β, A and $\pi^{i,j}$, $j \in \mathcal{N}^i$

Output: $\bar{\Omega}^i$ and Ω^i
- $\tilde{\Omega}^i = C^{iT} C^i$
- $l = 0$
- **while** $l < \bar{k}$ **do**
  - Receive $\bar{\Omega}^j$, $j \in \mathcal{N}^i$ from in-neighbours
  - $\tilde{\Omega}^i = \beta \left(A^{-1}\right)^T \left(\sum_{j \in \mathcal{N}^i} \pi^{i,j} \tilde{\Omega}^j \right) A^{-1}$
  - Send $\tilde{\Omega}^i$ to out-neighbours
  - $l = l + 1$
- **end while**
- $\bar{\Omega}^i := \beta \left(A^{-1}\right)^T \tilde{\Omega}^i A^{-1}$
- $\Omega^i := C^{iT} C^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}^j$

As noted in [13], for the distributed observer, the matrices $\pi^{i,j} A \left(\Omega^i\right)^{-1} \bar{\Omega}^j$ and $A \left(\Omega^i\right)^{-1} \left(C^i\right)^T \left(R^i\right)^{-1}$ for $i \in \mathcal{N}$ and $j \in \mathcal{N}^i$ can be precomputed offline, and therefore the online computations consist of $\left|\mathcal{N}^i\right| n^2 + n m_i$ multiplications and $n \left(\left|\mathcal{N}^i\right| n + m_i - 1 \right)$ sums.

4 Stability analysis

Defining the local estimation error as $e_t^i := \hat{x}_t^i - x_t$, from (1), (10), (11), and the fact that Π is stochastic, we obtain

$$e_{t+1}^i = A_t \left(\Omega_t^i\right)^{-1} \bar{e}_t^i, \tag{27}$$

where $\bar{e}_t^i := \sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j e_t^j$.

From Lemma 2 of [2], we obtain the following covariance intersection result

$$\bar{e}_t^{iT} \left(\sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j \right)^{-1} \bar{e}_t^i \leq \sum_{j \in \mathcal{N}} \pi^{i,j} e_t^{jT} \bar{\Omega}_t^j e_t^j. \tag{28}$$

From the definition of $G_{t,\tau}^i$ in (8), of $\bar{\Omega}_t^i$ in (12) and Ω_t^i in (11), it follows that

$$\beta A_t^{-1T} \Omega_t^i A_t^{-1} = \bar{\Omega}_{t+1}^i + \beta^{k+2} \mathcal{G}_{t,k+1}^i, \tag{29}$$

and noting that $\mathcal{G}^i_{t,k+1}$ is positive semidefinite, we have that

$$\bar{\Omega}^i_{t+1} \preceq \beta A_t^{-1T} \Omega^i_t A_t^{-1}. \tag{30}$$

We now present the main result of this paper.

Theorem 1 *Consider the distributed LTV observer* (10), *with matrices* Ω^i_t, *and* $\bar{\Omega}^i_t$ *computed as in* (11) *and* (13) *respectively. Given assumptions A1-A3, the estimation errors* $\hat{x}^i_t - x_t, i \in \mathcal{N}$ *converge exponentially to zero at a rate* β.

Proof From Assumption A4 and (12) one has that $\bar{\Omega}^i_t \succ \beta^{k+1}\alpha I_n$, for all i and t. Therefore we may define the local Lyapunov candidate function $\mathcal{L}^i_t := e^{iT}_t \bar{\Omega}^i_t e^i_t$. From (27), (28) and (30) we obtain

$$\begin{aligned}
\mathcal{L}^i_{t+1} &= \left(\bar{e}^i_t\right)^T \left(\Omega^i_t\right)^{-1} A_t^T \bar{\Omega}^i_{t+1} A_t \left(\Omega^i_t\right)^{-1} \left(\bar{e}^i_t\right) \\
&\leq \beta \left(\bar{e}^i_t\right)^T \left(\Omega^i_t\right)^{-1} \left(\bar{e}^i_t\right) \\
&\leq \beta \left(\bar{e}^i_t\right)^T \left(\sum_{j\in\mathcal{N}} \pi^{i,j} \bar{\Omega}^j_t\right)^{-1} \left(\bar{e}^i_t\right) \\
&\leq \beta \sum_{j\in\mathcal{N}} \pi^{i,j} e^{jT}_t \bar{\Omega}^j_t e^j_t = \beta \sum_{j\in\mathcal{N}} \pi^{i,j} \mathcal{L}^j_t.
\end{aligned}$$

In vector form, defining $\mathcal{L}_t := \mathrm{col}\left(\mathcal{L}^i_t\right)$ yields

$$\mathcal{L}_{t+1} \leq \beta \Pi \mathcal{L}_t, \tag{31}$$

where the inequality is interpreted element-wise.

Since Π is stochastic, 1 is an eigenvalue and we can find its left eigenvalue $\boldsymbol{p}$ which satisfies $\boldsymbol{p}^T\Pi = \boldsymbol{p}^T$. Finally, by defining the Lyapunov function $\mathcal{V}_t := \boldsymbol{p}^T\mathcal{L}_t$ we can compute

$$\mathcal{V}_{t+1} = \boldsymbol{p}^T\mathcal{L}_{t+1} \leq \beta \boldsymbol{p}^T \Pi \mathcal{L}_t = \beta \boldsymbol{p}^T \mathcal{L}_t = \beta \mathcal{V}. \tag{32}$$

Since the Lyapunov function decreases at each step, we have that the estimation errors converge to zero.

5 Numerical results

In this section, we illustrate the performance of the algorithm proposed in the paper through a design exercise. We also compare its performance against that obtained with the distributed Kalman filter algorithm with consensus on information in [3]. In the algorithm proposed in this paper, the parameter choice for β was 0.5, for α was 10^{-6} and for k 21.

To assess the performance of the distributed algorithms, we will consider a distributed system of the form (1)-(5) with collective but not local observability. We will consider a network of 20 nodes. The dynamical system considered has the state transition matrix defined as $A_t := (1 + d^A_t) I_N$, where d^A_t is a time-dependent random disturbance known to all the agents, which is normally distributed with a covariance of 10^{-2}.

Let e^i be a row vector with 1 at position i and zero at every other position. With this notation, the observation matrices are defined as

$$C^i := (1 + d^C_t) \begin{bmatrix} e^{iT} - e^{(i+1)T} \\ e^{(i-1)T} - e^{iT} \end{bmatrix},$$

where d^C_t is a time-dependent random disturbance known to all the agents, which is normally distributed with a covariance of 1, except at $i = 1$, where we replace $i - 1$ by N, and at $i = N$, where we define $C^N := (1 + d^C_t)e^{NT}$. This set of observation matrices translates to a setting where the heterogeneous systems with decoupled dynamics mentioned above have coupling in the measurements. It can be observed that with this choice of state transition and observation matrices, we have collective observability but not local observability at each node, thus requiring the use of distributed observers to reconstruct the state.

To simulate a more realistic situation process and measurement noise were added which were generated randomly with a Gaussian distribution. The covariances chosen for the disturbances were $Q = I_{2N}$ for process noise and $R^i = I_{m_i}$ for the measurement noise of each node. The initial state is also randomly generated with a Gaussian distribution with covariance $P_0 = 10^6 I_{2N}$. The communication network considered was an undirected circular network, i.e. the neighbour set at each node is defined as $\mathcal{N}^i := \{i-1, i+1\}$ except at node $i = 1$ where it is $\mathcal{N}^i := \{N, 2\}$, and at node $i = N$ where it is $\mathcal{N}^i := \{N-1, 1\}$.

In Figure 5 we compare the different algorithms in terms of the average of the norms of the estimation errors of all agents, i.e. $\frac{1}{N}\sum_{i\in\mathcal{N}} \left\|\hat{x}^i - x\right\|$. In the following plots we will present the results of a centralized Kalman filter (in blue) of the distributed Kalman filter in [3] (in red) and the algorithm of this paper (in yellow).

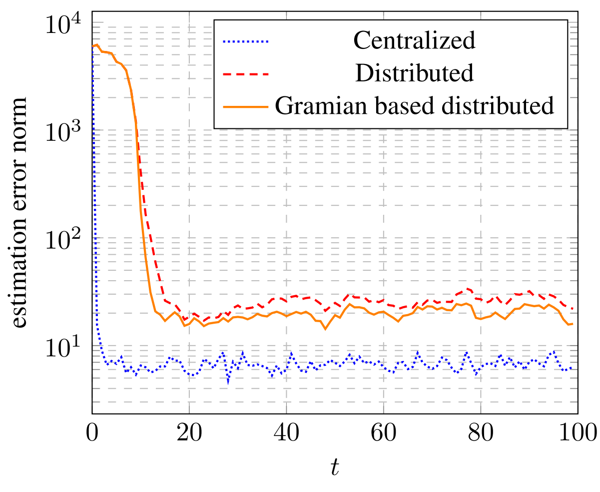

Figure 1. Average norm of the estimation errors with different methods.

From Figure 5 we can observe that the performance of the observer proposed in this paper is comparable to that of [3].

6 Conclusion

In this paper, an alternative method to design a distributed LTV observer with guaranteed stability is proposed, which requires the exchange of a matrix of size $l \times n$. It was shown, under the conditions of collective observability, strong connectivity of the sensor communication network, and invertibility of the state transition matrix that the resulting observer is stable. From the simulation results of an illustrative example, we showed that the asymptotic performance of the proposed observer is similar to that of [3]. In future work, I envision extending the proposed method in this paper to address the problem of distributed stabilization of LTV systems, possibly borrowing from the results in [9] for LTI systems.

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References

[1] Karl Aberer, Saket Sathe, Dipanjan Chakraborty, Alcherio Martinoli, Guillermo Barrenetxea, Boi Faltings, and Lothar Thiele. OpenSense: open community driven sensing of environment. In *Proceedings of the ACM SIGSPATIAL International Workshop on GeoStreaming - GIS '10*, pages 39–42, San Jose, California, 2010. ACM Press.

[2] Giorgio Battistelli and Luigi Chisci. Kullback-Leibler average, consensus on probability densities, and distributed state estimation with guaranteed stability. *Automatica*, 50(3):707–718, March 2014.

[3] Giorgio Battistelli, Luigi Chisci, Giovanni Mugnai, Alfonso Farina, and Antonio Graziano. Consensus-Based Linear and Nonlinear Filtering. *IEEE Transactions on Automatic Control*, 60(5):1410–1415, May 2015.

[4] Federico S. Cattivelli and Ali H. Sayed. Diffusion Strategies for Distributed Kalman Filtering and Smoothing. *IEEE Transactions on Automatic Control*, 55(9):2069–2084, September 2010.

[5] Álvaro Rodríguez del Nozal, Pablo Millán, Luis Orihuela, Alexandre Seuret, and Luca Zaccarian. Distributed estimation based on multi-hop subspace decomposition. *Automatica*, 99:213–220, January 2019.

[6] R. Ghabcheloo, A. P. Aguiar, A. Pascoal, C. Silvestre, I. Kaminer, and J. Hespanha. Coordinated Path-Following in the Presence of Communication Losses and Time Delays. *SIAM Journal on Control and Optimization*, 48(1):234–265, January 2009.

[7] Usman A. Khan and José M. F. Moura. Distributed Kalman Filters in Sensor Networks: Bipartite Fusion Graphs. In *2007 IEEE/SP 14th Workshop on Statistical Signal Processing*, pages 700–704, Madison, WI, USA, August 2007. IEEE.

[8] Usman A. Khan and José M. F. Moura. Distributing the Kalman Filter for Large-Scale Systems. *IEEE Transactions on Signal Processing*, 56(10):4919–4935, October 2008.

[9] Fengjiao Liu, Lili Wang, Daniel Fullmer, and A. Stephen Morse. Distributed Feedback Control of Multi-Channel Linear Systems, 2019. Publisher: arXiv Version Number: 6.

[10] Aritra Mitra and Shreyas Sundaram. An approach for distributed state estimation of LTI systems. In *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1088–1093, Monticello, IL, USA, September 2016. IEEE.

[11] Reza Olfati-Saber and Parisa Jalalkamali. Coupled Distributed Estimation and Control for Mobile Sensor Networks. *IEEE Transactions on Automatic Control*, 57(10):2609–2614, October 2012.

[12] Shinkyu Park and Nuno C. Martins. Design of Distributed LTI Observers for State Omniscience. *IEEE Transactions on Automatic Control*, 62(2):561–576, February 2017.

[13] Francisco F. C. Rego, A. Pedro Aguiar, António M. Pascoal, and Colin N. Jones. A design method for distributed luenberger observers. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 3374–3379, Melbourne, Australia, December 2017. IEEE.

[14] Francisco F. C. Rego, António M. Pascoal, A. Pedro Aguiar, and Colin N. Jones. Distributed state estimation for discrete-time linear time invariant systems: A survey. *Annual Reviews in Control*, 48:36–56, 2019.

[15] Joris Sijs and Paul Bosch. Heterogeneous state estimation in dynamic networked systems. *IET Control Theory & Applications*, 9(15):2232–2241, October 2015.

[16] Sayed Pouria Talebi, Sithan Kanna, Yili Xia, and Danilo P. Mandic. Cost-effective diffusion Kalman filtering with implicit measurement exchanges. In *2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4411–4415, New Orleans, LA, March 2017. IEEE.

[17] Yonggang Zhang, Chengcheng Wang, Ning Li, and Jonathon Chambers. Diffusion Kalman filter based on local estimate exchanges. In *2015 IEEE International Conference on Digital Signal Processing (DSP)*, pages 828–832, Singapore, Singapore, July 2015. IEEE.