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RESEARCH ARTICLE

The Robust Minimal Controllability and Observability Problem†

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Summary

In this paper, we study the minimal Robust Controllability and Observability Problem (rMCOP). The scenario that motivated this question is related to the design of a drone formation to execute some task, where the decision of which nodes to equip with more expensive communication system represents a critical economics choice. Given a linear time-invariant system for each of the vehicles, this problem consists in identifying a minimal subset of state variables to be actuated and measured, ensuring that the overall formation model is both controllable and observable, while tolerating a prescribed level of inputs/outputs that can fail.

Based on the tools in the available literature, a naive approach would consist in enumerating separately all possible minimal solutions for the controllability and observability parts. Then, iterating over all combination to find the maximum intersection of sensors/actuators in the independent solutions, yielding a combinatorial problem. The presented solution couples the design of both controllability and observability parts through a polynomial reformulation as a minimum set multi-covering problem under some mild assumptions. In this format, the algorithm has the following interesting attributes: i) only requires the solution of a single covering problem; ii) using polynomial approximations algorithms, one can obtain close-to-optimal solutions to the rMCOP.

KEYWORDS:

Robustness, Control design, Control applications, Minimal controllability and observability problem

1 | INTRODUCTION

Considering a Multi-Agent System (MAS) composed of vehicles interconnected by a communication network is a recurrent proposal for surveillance, exploration, and measuring tasks to be accomplished by unmanned and automatic robotic systems. Missions entailing the use of a large number of such vehicles can adopt a leader/followers approach characterized by having: i) expensive nodes (leaders) that can communicate with a ground station to receive mission commands and that might be equipped with sophisticated sensors or localization equipment: ii) cheaper drones (followers) implementing local controllers based on onboard sensors that measure relative localization and receive a small amount of data from the leaders. In this scenario depicted in Figure 1, a critical task is to minimize the number of leaders for economic reasons but without compromising the controllability and observability of the overall system. In distributed environments, these two properties are critical to be ensured in the systems [1,2]. We will refer to this challenge as the Minimal Controllability and Observability Problem (MCOP).

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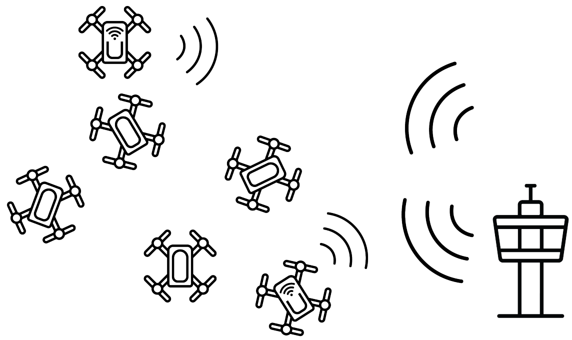

FIGURE 1 Depiction of the envisioned scenarios where the fixed tower represents the ground station, there are two expensive nodes (leaders) with the extra wireless symbol and four followers that use a controller based on local information related to nearby leaders.

In this paper, we focus on the subset where the nodes in the MCOP can be described by Linear Time-Invariant (LTI) models, which can be the result of a linearization of the original nonlinear model. Given that in a realistic environment there can be actuators or sensors that stop working either due to hardware faults, natural *phenomena* (e.g., due to the adverse nature of the environments where the actuators and sensors are placed) or by some software malfunctioning caused by an external entity (e.g., the scenario that took place in the Stuxnet malware incident [3]), we also study the robust version. In this case, the goal is to select a minimal number of leaders such that even if we have a prescribed number of input and output failures, the system is still controllable and observable, and we will refer it as the Robust Minimal Controllability and Observability Problem (rMCOP).

Related work

The controllability aspect of a dynamical system is dual to the observability of for linear systems [4]. In particular, Multi-Agent Systems (MAS) emerge in a plethora of areas, such as mathematics, biology, physics, sociology, and engineering applications [5,6,7,8], and can often be represented by a Linear Time-Invariant (LTI) or a Linear Parameter-Varying (LPV) system. The controllability of MAS having Laplacian dynamics was initially investigated in [9]. In [10,11], the authors found necessary and sufficient conditions in terms of partitions of the Laplacian graph for controllability. Paths and cycles were investigated in [12] and then extended to the controllability of grid graphs via reductions, symmetries, and scaled operations on the Laplacians [13]. In [14], the authors studied controllability and observability of MAS with heterogeneous and switching topologies, where the model equations for the position and velocity are different and switching in the network. Later, in [15], they studied the controllability and observability of switched multi-agent systems by constructing a switching sequence that ensures controllability, resorting to the concepts of the invariant subspace and the controllable state set. Necessary and sufficient conditions for both controllability and observability are also presented.

Structural systems allow us to efficiently design a minimal input or output placement for classes of LTI systems, by exploring the pattern of zeros and non-zeros of the dynamics matrix of the system. These are powerful tools for efficient design, ensuring almost surely the controllability (or observability) of the underlying system. Therefore, several frameworks have been developed in this scope. In [16], the authors present an efficient and unified framework to select the minimum number of manipulated/measured variables to reach the structural controllability/observability of the system. Also, it is provided a method to select the minimum number of feedback interconnections between measured and manipulated variables, ensuring the closed-loop system has no structural fixed modes. A model checking framework for LTI switching systems, using structural systems analysis, was presented in [17] and used in [18] to do the analysis and design of electric power grids with p-robustness guarantees, ensuring structural controllability

(that is, guaranteeing resilience to at most p transmission lines failures). However, the notions of structural controllability and structural observability are necessary, but not sufficient conditions to ensure systems' controllability and observability.

Solving the Minimal Controllability Problem (MCP) was shown to be NP-hard in [19]. In [20], the authors extended the MCP to address the robustness to inputs that may fail over time, showing that the Robust Minimal Controllability Problem (rMCP) is equivalent to a minimum set multi-covering problem, for which there exist efficient approximation algorithms with close-to-optimal guarantees. The authors extended the results for switched linear continuous-time systems in [21].

In this paper, for the first time, we address the problem of finding a minimal solution that ensures the system to be controllable and observable by affecting and measuring the smallest agent states. Since the previous works can be used to compute minimal solutions, ensuring controllability or observability separately, a simple approach consists of enumerating all possible solutions and select one with a maximal intersection, which would lead to a problem of combinatorial nature. In the remainder of this manuscript, we show that solving the MCOP is equivalent to a minimum set multi-covering problem.

Main contributions of this paper are the following: (i) we reduce the MCOP and the rMCOP to minimum set multi-covering problems; (ii) we show that almost all numerical instances of the input and output matrices that satisfy a specified structure yield solutions to the MCOP and rMCOP; (iii) we present a method that, given the structure of the input and the output matrices, produces numerical instances which are solutions to the rMCOP; (iv) we present illustrative examples of the main results.

The rest of the paper has the following structure. In Section 2, we present a formal description of the MCOP and rMCOP. Section 3 sets up the notation adopted in this work and the preliminary definitions that are basilar to the main results, which are presented in Section 4. In Section 5, we illustrate the main result of the manuscript, and we close the paper in Section 6, drawing conclusions and future research directions.

2 | PROBLEMS STATEMENT

An LTI system, under the adverse scenario of failure or a malicious entity tempering with system inputs and/or outputs, may be described as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B^{\mathcal{M}\setminus\mathcal{A}}u(t) \\ y(t) &= C^{\mathcal{N}\setminus\mathcal{A}}x(t) \end{aligned}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $x(0) = x_0$, $u(t) \in \mathbb{R}^p$ is a continuous input signal and $y(t) \in \mathbb{R}^q$ is a continuous output signal. Furthermore, $B^{\mathcal{M}\setminus\mathcal{A}}$ denotes the set of columns of B with indices in $\mathcal{M} \setminus \mathcal{A}$, where $\mathcal{M} = \{1, \dots, p\}$ is the set of input indices and $\mathcal{A}$ the set of indices of malfunctioning inputs/outputs. Analogously, $C^{\mathcal{N}\setminus\mathcal{A}}$ is the set of rows of C with indices in $\mathcal{N} \setminus \mathcal{B}$, where $\mathcal{N} = \{1, \dots, q\}$ is the set of output indices.

For convenience, let us refer to a system given in the format of (1) by the triple $(A, B^{\mathcal{M}\setminus\mathcal{A}}, C^{\mathcal{N}\setminus\mathcal{A}})$ and, when $\mathcal{A}$ is explicit from the context, we simply use the triple (A, B, C), to easy notation.

The first problem that we tackle is how to find two minimal sets — one for the states to be actuated and another for states to be measured — such that the intersection has maximum cardinality while keeping the overall system controllable and observable. It can be stated as the solution to the following optimization problem:

Problem 1 (MCOP)**.** Given a LTI system $\dot{x}(t) = Ax(t)$, $x(0) = x_0$ and $x \in \{0, 1\}^n$, find $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$ as a solution of the optimization problem:

$$\begin{aligned} (B^*, C^*) \;&=\; \underset{B,C\in\mathbb{R}^{n\times n}}{\arg\min}\; \|B\|_0 + \|C\|_0 - \|B \odot C\|_0, \\ \text{s.t.}\;& \;(A, B, C) \text{ is controllable and observable.} \end{aligned}$$

∘

Observe that we allow the matrices B and C to have sizes $n \times n$ to ensure that a solution always exists, i.e., picking $n \times n$ identity matrices ensures controllability and observability. The second problem addressed in this paper is the robust version of Problem 1. In other words, besides ensuring the system to be controllable and observable, we further want to guarantee the system to remain controllable and observable despite the existence of input or output failures.

Problem 2 (rMCOP)**.** Given a LTI system $\dot{x}(t) = Ax(t)$, $x(0) = x_0$ with $x \in \{0, 1\}^n$, and given the maximum number of inputs+outputs that may fail, $s \in \mathbb{N}$, find $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$ as a solution of the optimization problem:

$$\begin{aligned} (B^*, C^*) = \;& \underset{B,C\in\mathbb{R}^{n\times(s+1)n}}{\arg\min}\; \|B\|_0 + \|C\|_0 - \|B \odot C\|_0, \\ \text{s.t.}\;& \;(A, B^{\mathcal{M}\setminus\mathcal{A}}, C^{\mathcal{N}\setminus\mathcal{A}}) \text{ is controllable and observable} \\ & \;\mathcal{A} \subset \mathcal{M} \cup \mathcal{N} \text{ and } |\mathcal{A}| \leq s \end{aligned}$$

∘

Notice that, in this case, we allow the matrices B and C to have sizes $n \times (s + 1)n$ to ensure that a solution always exists, i.e., the concatenation of $(s + 1)$ times the $n \times n$ identity matrix always ensure that the system is controllable and observable even when s inputs+outputs fail.

Remark that Problem 1 is a particular case of Problem 2 where $s = 0$. In the remainder, we focus on Problem 2 and get as a byproduct the solution to Problem 1.

The following assumptions are needed for the adopted solution.

Assumption 1. The dynamics matrix A is diagonalizable.

Note that Assuption 1 is not very restrictive, because there are several applications where this assumption holds. Namely, dynamical systems that are modeled by random networks of the Erdös-Rényi type[22]; and popular benchmark dynamical systems utilized in control systems engineering[23,24].

Assumption 2. A *left-eigenbasis* and a *right-eigenbasis* of A are available, i.e., we have access to the left-eigenvectors and right-eigenvectors.

Assumption 2 is technically needed although, in practice, we may only compute the eigenvectors up to a certain precision.

3 | PRELIMINARIES & TERMINOLOGY

We denote the $n \times n$ identity matrix by I_n. We denote sets of numbers by calligraphic letters, e.g., $\mathcal{I}, \mathcal{S}, \mathcal{U}, \mathcal{J}$. We denote by $I_n(\mathcal{I})$, where $\mathcal{I} \subseteq \{1, \dots, n\}$, the $n \times n$ matrix with the columns with indices in $\mathcal{I}$ equal to the columns of I_n and the remaining columns equal to zero. Analogously, given a matrix $B \in \mathbb{R}^{n\times m}$ and a set $\mathcal{M} \subset \{1, \dots, m\}$, we denote by $B(\mathcal{M}) \in \mathbb{R}^{n\times m}$ the matrix composed by the columns of B with indices in $\mathcal{M}$ and the remaining columns equal to zero. We denote by $\mathbf{0}_{n,m}$ the $n \times m$ matrix of zeros and, similarly, by $\mathbf{1}_{n,m}$ the $n \times m$ matrix of ones. Further, when the dimensions are obvious from the context, we omit the dimensions and write $\mathbf{0}$ and $\mathbf{1}$. Given a square matrix A we denote its *spectrum* by $\sigma(A)$, i.e., the set of eigenvalues of A.

We will use the Popov-Belevitch-Hautus (PBH) eigenvector controllability and observability tests. Consider an LTI system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ and $x(0) = x_0$, with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$. The PBH eigenvector test for controllability states that the system is controllable if $v^\intercal B \neq 0$ for each left-eigenvector v of A. Similarly, the PBH eigenvector test for observability states that the system is observable if $Cu \neq 0$ for each right-eigenvector u of A.

Definition 1 (Minimum Set Covering Problem[25])**.** Given a universe of m elements $\mathcal{U}$, a collection of n set $\{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, with $\mathcal{S}_i \subseteq \mathcal{U}$, for $i \in \{1 \dots, n\}$, such that $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$. The minimum set covering problem consists in finding a set of indices $\mathcal{J}^* \subseteq \{1, \dots, n\}$ such that $\bigcup_{i\in\mathcal{J}^*} \mathcal{S}_i = \mathcal{U}$. That is,

$$\mathcal{J}^* = \underset{\mathcal{J}\subseteq\{1,\dots,n\}}{\arg\min}\ |\mathcal{J}|. \tag{2}$$

A generalization of the previous problem consists in requiring each element of the universe to be covered, at least, a specified number of times. This extension has the following definition.

Definition 2 (Minimum Set Multi-covering Problem[26])**.** Given a universe of m elements $\mathcal{U}$, a collection of n sets $\{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, with $\mathcal{S}_i \subseteq \mathcal{U}$, for $i \in \{1 \dots, n\}$, such that $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$, and a demand function $d : \mathcal{U} \to \mathbb{N}$ indicating the number of times an element u has to be covered. In other words, $d(u)$ is the minimum number of sets that containing element u that need to be considered. The minimum set multi-covering problem consists in finding a set of indices $\mathcal{J}^* \subseteq \{1, \dots, n\}$ such that $\bigcup_{i\in\mathcal{J}^*} \mathcal{S}_i = \mathcal{U}$ and every element $u \in \mathcal{U}$ is covered $d(u)$ times. That is,

$$\begin{aligned} \mathcal{J}^* = \underset{\mathcal{J}\subseteq\{1,\dots,n\}}{\arg\min}\ & |\mathcal{J}| \\ \text{s.t. } & |\{i \in \mathcal{J} : u \in \mathcal{S}_i\}| \geq d(u). \end{aligned} \tag{3}$$

The previous problems will play a key role in our proposal for Problems 1 and 2.

4 | MAIN RESULTS

In this section, we investigate solutions to Problems 1 and 2 by rewriting them as in minimum set multi-covering problems. To that end, we present the following algorithm.

Algorithm 1 Polynomial reduction of the structural optimization Problem 2, to a set multi-covering problem

Input: $\left\{\bar{v}^j\right\}_{j\in\mathcal{J}}$ and $\left\{\bar{u}^j\right\}_{j\in\mathcal{K}}$, two collections each of n vectors, both in $\{0,\star\}^n$ and $s\in\mathbb{N}$, the maximum number of inputs and outputs that may fail.

Output: $\mathcal{S}=\left\{\mathcal{S}_j\right\}_{j\in\{1,\ldots,(s+1)n\}}$ and $\mathcal{U}$, a set with n sets, and the universe of these sets, respectively.

1: $\mathcal{S}_j=\emptyset$, for $j\in\{1,\ldots,(s+1)n\}$

2: **for** $j=1,\ldots,|\mathcal{J}|$
  for $k=1,\ldots,n$
    if $[\bar{v}^j]_k\neq 0$ **then**
      $\mathcal{S}_k=\mathcal{S}_k\cup\{j\}$

3: **for** $j=1,\ldots,|\mathcal{K}|$
  for $k=1,\ldots,n$
    if $[\bar{u}^j]_k\neq 0$ **then**
      $\mathcal{S}_k=\mathcal{S}_k\cup\{|\mathcal{J}|+j\}$

4: **for** $l=1,\ldots,s-1$
  $\mathcal{S}_{ln+k}=\mathcal{S}_k$

5: **set** $\mathcal{S}=\left\{\mathcal{S}_j\right\}_{j\in\{1,\ldots,(s+1)n\}}$, $\mathcal{U}=\bigcup\limits_{\mathcal{V}\in\mathcal{S}}\mathcal{V}$ and $d(i)=s+1$ for $i\in\mathcal{U}$.

Lemma 1. Given two collections of non-zero vectors $\{\bar{v}^j\}_{j\in\mathcal{J}}$ and $\{\bar{u}^k\}_{k\in\mathcal{K}}$, with $\bar{v}^j,\bar{u}^k\in\{0,1\}^n$, and $s\in\mathbb{N}$, finding $\bar{B}^*\in\{0,1\}^{n\times n}$ and $\bar{C}^*\in\{0,1\}^{n\times n}$ such that

$$\begin{aligned}\bar{B}^*,\bar{C}^* = \ &\underset{\bar{B},\bar{C}\in\{0,1\}^{n\times n}}{\arg\min}\ \|\bar{B}\|_0+\|\bar{C}\|_0-\|\bar{B}\odot\bar{C}\|_0\\ \text{s.t. } &\bar{v}^j\cdot\bar{B}(\mathcal{M})\neq 0 \text{ and}\\ &\bar{u}^k\cdot\bar{C}(\mathcal{M})\neq 0,\ \text{ for all } j\in\mathcal{J} \text{ and } k\in\mathcal{K},\\ &\text{where } \mathcal{M}\subset\{1,\ldots,n\} \text{ and } |\mathcal{M}|\geq n-s.\end{aligned}\tag{4}$$

is polynomially reducible, in $|\mathcal{J}|$, $|\mathcal{K}|$ and n, to a minimum set multi-covering problem with universe $\mathcal{U}$, collection of sets $\mathcal{S}$ and demand function d by applying Algorithm 1.

Proof. Let $\mathcal{S}$ be a collection of sets, $\mathcal{U}$ a universe and d a demand function that result from Algorithm 1. Then, we have the following equivalences. Let $\mathcal{I}\subset\{1,\ldots,n,\ldots,(s+1)n\}$ be a set of indices. Further, let $\mathcal{I}^b=\{i\ :\ i\in\mathcal{I}\text{ and } i\leq n\}$, and let $\bar{B}\equiv\bar{B}(\mathcal{I}^b)\in\{0,1\}^{n\times(s+1)n}$ be a structural matrix such that $[\bar{B}]_{j,i}\neq 0$, with $j=(i\mod n)+1$, if and only if $i\in\mathcal{I}^b$. Analogously, let $\mathcal{I}^c=\{i\ :\ i\in\mathcal{I}\text{ and } i>n\}$, and let $\bar{C}\equiv\bar{C}(\mathcal{I}^c)\in\{0,1\}^{n\times(s+1)n}$ be a structural matrix such that $[\bar{C}]_{j,i}\neq 0$, with $j=(i\mod n)+1$, if and only if $i\in\mathcal{I}^c$. Observe that $\mathcal{I}=\mathcal{I}^b\cup\mathcal{I}^c$ and $\mathcal{I}^b\cup\mathcal{I}^c=\emptyset$. Then, since $\mathcal{I}$ is a solution of the set multi-covering problem, we have that the collection of sets $\{\mathcal{S}_i\}_{i\in\mathcal{I}}$ covers $\mathcal{U}$ and satisfies the demand function d if and only if $\forall j\in\mathcal{J}\ \exists l\in\mathcal{I}$ such that $j\in\mathcal{S}_l$ and $|\{r\in\mathcal{I}\ :\ j\in\mathcal{S}_r\}|\geq d(j)=s+1$. This is equivalent to the conjunction of the twofold:

(*i*) $\forall j\in\mathcal{J}\ \exists l\in\mathcal{I}^b$ such that $\bar{v}^j_l\bar{B}_l\neq\mathbf{0}$ and $|\{r\ :\ \bar{v}^j_r\bar{B}_r\neq\mathbf{0}\}|\geq d(j)=s+1$;

(*ii*) $\forall k\in\mathcal{K}\ \exists l\in\mathcal{I}^c$ such that $\bar{u}^k_l\bar{C}_l\neq\mathbf{0}$ and $|\{r\ :\ \bar{u}^k_l\bar{B}_l\neq\mathbf{0}\}|\geq d(k)=s+1$;

This is equivalent to $\forall j\in\mathcal{J}\ \bar{v}^j\cdot\bar{B}\neq\mathbf{0}$ and $|\{l\ :\ v^j\cdot\bar{B}_l\neq 0\}|\geq s+1$, and, similarly, $\forall k\in\mathcal{K}\ \bar{u}^k\cdot\bar{C}\neq\mathbf{0}$ and $|\{l\ :\ u^k\cdot\bar{C}_l\neq 0\}|\geq s+1$. Hence, even if s entries of $\bar{B}$ become zero (s inputs fail) it is still true that $\bar{v}^j\cdot\bar{B}\neq 0$. Analogously, if s entries of $\bar{B}$ become zero (s ouputs fail) it is still true that $\bar{u}^k\cdot\bar{C}\neq 0$.

In summary, $(\bar{B},\bar{C})$ is a feasible solution to Problem 2. Moreover, we observe that the reduction of Problem 2 to a minimum set multi-covering problem produces a sparsity pattern $(\bar{B},\bar{C})$ that is the sparsity of a solution to Problem 2. Finally, the cost of

Algorithm 1 is $\mathcal{O}(n^2.\max\{|\mathcal{J}|,|\mathcal{K}|\})$ and, as envisaged, the aforementioned reduction has polynomial time and space complexity in $|\mathcal{J}|$, $|\mathcal{K}|$ and n. □

In fact, given the structure of a feasible solution of Problems 1 or 2, any numerical realization leads to a solution for the problem.

Now, building upon Lemma 1, we state the main result of this paper.

Theorem 1. The rMCOP can be solved in two steps:

(i) identifying the sparsity of the input and output matrices, $(\bar{B},\bar{C})$, using Lemma 1;

(ii) choosing a numerical realization for $(\bar{B},\bar{C})$.

Proof. By assumption 2 a left and a right-eigenbasis are available and by assumption 1 each eigenbasis is composed by n vectors. Hence, the proof follows by noticing that Lemma 1 produces a feasible solution to the rMCOP that satisfies the PHB eigenvector test for controllability and observability, that can be used to find a numerical solution. □

Further, we observe that to solve a rMCOP (or a MCOP) is computationally demanding. In fact, we have the following.

Theorem 2. Both the MCOP and the rMCOP are NP-hard.

Proof. The proof follows by noticing that a subproblem of the MCOP is the minimal controllability problem (MCP) and a subproblem of the rMCOP is the robust minimal controllability problem (rMCP), i.e., when we only seek to find an input matrix, and both problems are NP-hard[20], that is when one of the input collections of vectors of Algorithm 1 is empty. □

Next, we illustrate the main results with examples.

5 | ILLUSTRATIVE EXAMPLES

In the following, we illustrate the use of the proposed methods with synthetic and real world examples.

5.1 | Synthetic examples

Consider the linear system with dynamics matrix

$$A=\frac{1}{2}\begin{bmatrix} 10 & 0 & 0 & 2 & 2\\ 3 & 6 & 3 & 2 & 1\\ -3 & 0 & 3 & -4 & -3\\ 3 & 0 & -1 & 6 & 3\\ -3 & 0 & 1 & 2 & 5\end{bmatrix}$$

The eigenvalues of A are $\sigma=\{1,2,3,4,5\}$ and hence the matrix is simple. The left-eigenvectors of A are

$$V^L=\begin{bmatrix} | & | & | & | & | \\ v_1^L & v_2^L & v_3^L & v_4^L & v_5^L \\ | & | & | & | & | \end{bmatrix}=\begin{bmatrix} 1&0&0&1&0\\ 0&0&1&0&0\\ 0&0&1&1&1\\ 1&1&1&0&1\\ 1&1&1&1&0\end{bmatrix}$$

Further, the right-eigenvectors of A are

$$V^R=\begin{bmatrix} | & | & | & | & | \\ u_1^R & u_2^R & u_3^R & u_4^R & u_5^R \\ | & | & | & | & | \end{bmatrix}=\begin{bmatrix} -2&-1&0&0&0\\ -1&-1&1&-1&1\\ 1&0&0&1&-1\\ -1&0&0&-1&-1\\ 1&1&0&1&1\end{bmatrix}$$

We start by addressing Problem 1. To illustrate the main results, we follow two approaches:

(i) find, independently, a minimal solution to ensure controllability and a minimal solution to ensure observability with the maximum number of common state variables to actuate and observe;

(ii) use Algorithm 1 to solve the MCOP (rMCOP with $s = 0$).

Approach (i)

If we use the method in[20] to find a solution to the minimal controllably problem, we build the sets

$$\begin{array}{rl} \mathcal{S}_1^1 = \{1,4\} & \leftarrow \\ \mathcal{S}_2^1 = \{3\} & \leftarrow \\ \mathcal{S}_3^1 = \{3,4,5\} & \leftarrow \\ \mathcal{S}_4^1 = \{1,2,3,5\} & \leftarrow \\ \mathcal{S}_5^1 = \{1,2,3,4\} & \leftarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = V^L.$$

This sets constitute the universe $\mathcal{U} = \bigcup_{i=1}^5 \mathcal{S}_i^1 = \{1,\dots,5\}$. By solving the associated minimum set covering problem, the possible solutions are $\mathcal{I}_1^1 = \{1,4\}$ ($\mathcal{U} = \mathcal{S}_1^1 \cup \mathcal{S}_4^1$), $\mathcal{I}_2^1 = \{3,4\}$ ($\mathcal{U} = \mathcal{S}_3^1 \cup \mathcal{S}_4^1$), $\mathcal{I}_3^1 = \{3,5\}$ ($\mathcal{U} = \mathcal{S}_3^1 \cup \mathcal{S}_5^1$) and $\mathcal{I}_4^1 = \{4,5\}$ ($\mathcal{U} = \mathcal{S}_4^1 \cup \mathcal{S}_5^1$).

Analogously, by invoking the duality between controllability and observability, we can use[20] to solve the dual problem, the minimal observability problem. In this case, we build the sets

$$\begin{array}{rl} \mathcal{S}_1^2 = \{1,2\} & \leftarrow \\ \mathcal{S}_2^2 = \{1,2,3,4,5\} & \leftarrow \\ \mathcal{S}_3^2 = \{1,4,5\} & \leftarrow \\ \mathcal{S}_4^2 = \{1,4,5\} & \leftarrow \\ \mathcal{S}_5^2 = \{1,2,4,5\} & \leftarrow \end{array} \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} = V^R.$$

Again, this sets constitute the universe $\mathcal{U} = \bigcup_{i=1}^5 \mathcal{S}_i^2 = \{1,\dots,5\}$. Solving the associated minimum set covering problem, the possible solution is $\mathcal{I}_1^2 = \{2\}$. Therefore, the pairs of solutions to both problems with maximum intersection are: $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, $(\mathcal{I}_2^1, \mathcal{I}_1^2)$, $(\mathcal{I}_3^1, \mathcal{I}_1^2)$ and $(\mathcal{I}_4^1, \mathcal{I}_1^2)$. All the cases result in input matrices and output matrices where $\|B\|_0 + \|C\|_0 - \|B \odot C\|_0 = 2 + 1 - 0 = 3$. For instance, for the pair of solutions $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, we obtain the following

$$B(\mathcal{I}_1^1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$C(\mathcal{I}_1^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $v_i^L \cdot B(\mathcal{I}_1^1) \neq 0$ and $u_i^R \cdot C(\mathcal{I}_1^2) \neq 0$ for any $i = 1,\dots,n$. Thus, by the PBH eigenvector criteria for controllability and observability, the system (A, B, C) is controllable and observable.

Approach (ii)

Using Algorithm 1, we have that

$$\begin{array}{rl} \mathcal{S}_1 = \{1,4,6,7\} & \leftarrow \\ \mathcal{S}_2 = \{3,6,7,8,9,10\} & \leftarrow \\ \mathcal{S}_3 = \{3,4,5,6,9,10\} & \leftarrow \\ \mathcal{S}_4 = \{1,2,3,5,6,9,10\} & \leftarrow \\ \mathcal{S}_5 = \{1,2,3,4,6,7,9,10\} & \leftarrow \end{array} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{matrix}}_{V^L} & \underbrace{\begin{matrix} -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{matrix}}_{V^R} \end{array}\right].$$

Thus, the resulting universe is $\mathcal{U} = \bigcup_{i=1}^{5} \mathcal{S}_i = \{1, \dots, 10\}$. By solving the associated minimum set covering problem, we obtain as solution $\mathcal{I}_1 = \{1, 2, 4\}$ (or $\mathcal{I}_2 = \{2, 4, 5\}$). This solution translates to assign an input to each variable in $\mathcal{I}_1^1 = \{1, 2, 4\}$ (or $\mathcal{I}_2^1 = \{2, 4, 5\}$) and to assign an output variable to each variable in $\mathcal{I}_1^2 = \{1, 2, 4\}$ (or $\mathcal{I}_2^2 = \{2, 4, 5\}$). This scenario results in input matrices and output matrices such that $\|B\|_0 + \|C\|_0 - \|B \odot C\|_0 = 3 + 3 - 3 = 3$. For instance, for the solution $\mathcal{I}_1$, we obtain the following

$$B(\mathcal{I}_1) = C(\mathcal{I}_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 1. We draw the attention of the reader to the fact that, in approach (i), we had to enumerate all solutions $\mathcal{I}_1^1 \cdots \mathcal{I}_4^1$ using a polynomial approximation algorithm (similar for the observability). Then, one computes the cost for each combination of solutions, leading to a computationally expensive algorithm. Although both approaches found solutions to the MCOP, the approach (ii) only requires solving one minimum set covering problem. In other words, the approach we propose in this paper (approach (ii)) is much less computationally demanding than the naïve approach (approach (i)).

Next, we explore the previous example in the robust scenario stated in Problem 2. In the scenario where one input+output may fail, $s = 1$, we solve the associated Problem 2 using Algorithm 1. We obtain the universe $\mathcal{U} = \{1, \dots, 10\}$, demand function $d(i) = 2$ for $i \in \mathcal{U}$ and the following sets $\mathcal{S}_1 = \mathcal{S}_6 = \{1, 4, 6, 7\}$, $\mathcal{S}_2 = \mathcal{S}_7 = \{3, 6, 7, 8, 9, 10\}$, $\mathcal{S}_3 = \mathcal{S}_8 = \{3, 4, 5, 6, 9, 10\}$, $\mathcal{S}_4 = \mathcal{S}_9 = \{1, 2, 3, 5, 6, 9, 10\}$ and $\mathcal{S}_5 = \mathcal{S}_{10} = \{1, 2, 3, 4, 6, 7, 9, 10\}$. By solving the associated set multi-covering problem, we obtain as solution $\mathcal{I} = \{2, 3, 4, 5, 7\}$, which translates into assigning one input and one output to each state variable in $\{2, 3, 4, 5, 2\}$. Notice that the obtained solution cannot be achieved by picking 2 solutions for Problem 1, i.e., in general, a minimal rMCOP solution cannot be achieved by "stacking" s solutions to the MCOP. In this example, using 2 sets from the MCOP would require a union of state variables to observe and control with cardinality equal to 6, whereas the solution to the rMCOP achieves the same with 5 variables, and assigning inputs to these solutions. By doing so, we would need to control and observe 6 state variables, instead of 5, and the solution would not be minimal.

5.2 | Real world example

Consider a mass-spring-damper oscillator with the following dynamics[27]:

$$\dot{x}_o(t) = A_o x_o(t), \text{ where } A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix},$$

with $x_o(0) = x_0$, and where ω_n is the natural frequency and ζ is the damping ratio. Consider two mass-spring-damper oscillator interconnected by the following dynamics.

$$\dot{x}(t) = Ax(t),$$

where

$$A = \left[\begin{array}{cc|cc} 0 & 1 & -1 & 0 \\ -\omega_n^2 & -2\zeta\omega_n & 0 & 1 \\ \hline -1 & 0 & 0 & 1 \\ 0 & 1 & -\omega'^2_n & -2\zeta'\omega'_n \end{array}\right],$$

with $\omega_n = 2$ Hz, $\omega'_n = 3$ Hz, $\zeta = \zeta' = 0$ (both oscillators are undamped) and $x(0) = (x_0, x'_0)$. The eigenvalues of A are $\sigma = \left\{2i\sqrt{2}, -2i\sqrt{2}, i\sqrt{3}, -i\sqrt{3}\right\}$ and hence the matrix is simple. The left-eigenvectors of A are

$$V^L = \begin{bmatrix} | & | & | & | \\ v_1^L & v_2^L & v_3^L & v_4^L \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} -1 & -1 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -1 & -1 \\ -\frac{\sqrt{7}}{3} & \frac{\sqrt{7}}{3} & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

and, the right-eigenvectors of A are

$$V^R = \begin{bmatrix} | & | & | & | \\ u_1^R & u_2^R & u_3^R & u_4^R \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & i\sqrt{3} & -i\sqrt{3} \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Consider that we want solve the associated rMCOP problem with $s = 1$. Using Algorithm 1, we have that

$$\begin{array}{r} \mathcal{S}_1 = \{1,2,3,4,7,8\} \leftarrow \\ \mathcal{S}_2 = \{3,4,5,6,7,8\} \leftarrow \\ \mathcal{S}_3 = \{1,2,3,4,5,6\} \leftarrow \\ \mathcal{S}_4 = \{1,2,5,6,7,8\} \leftarrow \end{array} \left[\underbrace{\begin{matrix} -1 & -1 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -1 & -1 \\ -\frac{\sqrt{7}}{3} & \frac{\sqrt{7}}{3} & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}}_{V^L} \middle| \underbrace{\begin{matrix} 0 & 0 & 1 & 1 \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & i\sqrt{3} & -i\sqrt{3} \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}}_{V^R} \right]$$

the universe of the minimum set multi-covering problem is $\mathcal{U} = \{1, \ldots, 8\}$ and $d(i) = s + 1 = 2$ for each $i \in \mathcal{U}$. By solving this minimum set multi-covering problem, we obtain as a possible solution $\mathcal{I} = \{1, 2, 3\}$. Thus, one could use numerical matrices:

$$B(\mathcal{I}) = C(\mathcal{I}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that any two non-zero columns of B (or of C) are not orthogonal to the left-eigenvectors (or right-eigenvectors) of A. Hence, by the PBH eigenvectors test for controllability and observability, the system is controllable and observable even when one of the inputs or outputs fail. Further, notice that if $s = 0$, a solution would have to consider two of the sets (e.g. $\mathcal{S}_1$ and $\mathcal{S}_2$). Nonetheless, considering two solutions for the scenario with $s = 0$ would not lead to an optimal number of actuated and measured state variables.

6 | CONCLUSIONS

In this paper, we have addressed the Robust Minimal Controllability and Observability Problem (rMCOP), given the motivation of selecting how many leaders are needed in a Multi-Agent System (MAS) for economic reasons. The problem consists of identifying a small number of state variables to be actuated and observed that ensures the system to be both controllable and observable when a specified maximum number of inputs and outputs may fail over time. The naïve solution using the tools in the literature would be to decouple the controllability and observability parts, resulting in a combinatorial solution. However, through an integrated analysis of the two components, we can reduce the task to the computation of a solution to the minimum set multi-covering problem. Consequently, we may either explicitly solve a set multi-covering problem and obtain the optimal solution to the rMCOP (combinatorial), or approximate the solution resorting to efficient algorithms with close-to-optimality guarantees. We envision as future research considering the same problem for networks evolving in discrete time. Other directions of interest include exploring the case of non-simple dynamics matrices or adding a cost function to account for the economics or restrictions associated with the mission to be carried by the MAS.

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