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A Novel and Efficient Order Reduction for both Constrained Convex Generators and Constrained Zonotopes

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***Abstract*— A central challenge with any reachability technique is the growth over time of the data structures that store the set-valued estimates. There are various techniques established for Constrained Zonotopes (CZ), although their computational complexity represents a limiting factor on the size of the set descriptions when running the methods in real-time. Thus, when running a guaranteed state observer to estimate the state of a dynamical system using CZs, the number of generators and constraints has to be maintained small such that the order reduction procedures can be run within the sampling time. This paper resorts to using ellipsoids for portions of the set description, which results in a computationally efficient method for a particular class of constrained Convex Generators (CCGs) that can also be used for ellipsotopes and CZs. Our approach is shown to have comparable performance and in some cases outperforms existing methods for Constrained Zonotopes. We provide numerical examples to illustrate the advantages of our proposed approach, particularly in the context of guaranteed state estimation.**

***Index Terms*— Order Reduction; Constrained Convex Generators; Reachability analysis.**

I. Introduction

The problem of efficiently approximating convex sets has a wide range of applications, including state estimation, collision avoidance for autonomous vehicles, safe optimal control with Control Barrier Functions (CBFs), and certification of neural network-based controllers [1]–[6]. One of the main challenges with convex set representations is obtaining methods that are both computationally efficient and do not excessively over-approximate the set of admissible states, while also avoiding the wrapping effect caused by over-approximations. To address this challenge, it is necessary to undertake order reduction procedures that will find an over-approximation of the set-valued estimate that can be represented by a smaller data structure.

Several approaches can be found in the literature for an approximate representation of convex sets. One possible approach for convex set representation is using ellipsoids, as in [7], [8] and [9]. Another possibility is using zonotopes, as in [10], [11], and [12], which are shown to have reduced wrapping effect [13]. More recently, constrained zonotopes (CZs) have been gaining attention [14]. Ellipsotopes [15] provide a unified representation of these two set representations. An even more general representation is constrained convex generators (CCGs), proposed in [16], which we consider in this paper.

For any type of representation, in order to represent more complex shapes larger data structures are required. However, for practical applications with limited computational resources, there is a limit on the amount of data one can use to represent a set. Therefore, in practice, it is often necessary to reduce the complexity of the sets by computing an overapproximation. This process is named order reduction and in the literature, one can find multiple examples associated with specific set representations, such as zonotopes [11], [17], CZs [14], or ellipsotopes [15].

The need for an efficient order reduction is paramount in case the dynamical equation has uncertain parameters. Alternative formulations which mitigate the increase in set complexity include pointwise operations (like in [18]) or defining over-approximations for the product between an interval matrix and the desired set representation as it was done to handle nonlinear dynamics in [19] for CZs. However, the former type will not be able to handle large state space and the latter method is going to be very conservative. The most accurate approach will resort to the computation of the convex hull for propagation using the dynamics for each vertex of the uncertain space. However, the convex hull operation in guaranteed state estimation adds an exponential growth in the generators and constraints for CZs and CCGs alike (see [20] for CZs and [21] for CCGs). Even the recent introduction of the optimal representation of convex hulls for CCGs in [22] still reports the need for an efficient order reduction. Thus, the current proposal is instrumental for an accurate and efficient method for state estimation of uncertain Linear Parameter-Varying (LPV) systems.

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In this work, we propose a computationally efficient method for order reduction of CCGs using ellipsoids. In certain cases, our approach outperforms existing methods, including classical methods of constrained zonotope order reduction. We provide numerical examples to illustrate the advantages of our proposed approach and demonstrate its effectiveness in the context of guaranteed state estimation. In particular, the contributions of the paper are the following:

- Based on the methods proposed in [14] and [15], we propose an order and constraint reduction method for a class of CCGs, that is similar but more general than ellipsotopes.
- We propose an event-triggered guaranteed state estimation mechanism for real-time applications.
- We illustrate the performance of the proposed methods with numerical results.

This paper is organized as follows. Section II introduces the mathematical background required for the rest of the paper. Section III describes the proposed methods of order reduction, constraint reduction and event-triggered guaranteed state estimation. Section IV contains the numerical results comparing the performance of the proposed method with classical order reduction methods of CZs. Finally, Section V contains the main conclusions from this paper.

A. Notation

Let I_n be the identity matrix of size n, and let $\mathbf{0}_n$ stand for the n-dimensional array of zeros and $\mathbf{1}_n$ denote the n-dimensional array of ones. Whenever the index is omitted $\mathbf{0}$ denotes a matrix of zeros whose size can be inferred from the context. The transpose of a vector v is written by $v^\intercal$, and the Euclidean norm for a vector x is denoted by $\|x\|_2 := \sqrt{x^\intercal x}$. Additionally, $\|x\|_\infty := \max_i |x(i)|$, where $x(i)$ is the ith element of x. The generalised intersection is represented by $\cap_R$, the Minkowski sum of two sets by $\oplus$, and the cartesian product by $\times$. Given a matrix X, the operator $\operatorname{null}(X)$ gives an orthonormal basis of the null-space of X, $X^\dagger$ is the pseudo-inverse of X and $\det(X)$ is the determinant of X. Given a set of matrices X_i, $i \in \{1, \ldots, n\}$ $\operatorname{diag}(X_i)$ yields a block diagonal matrix whose diagonal blocks are X_i.

II. Mathematical Background

We first introduce the definition and the main operations of CCGs. Definition 1 provide a formal description of CCGs.

Definition 1 (Constrained Convex Generators): $\mathcal{Z} \subset \mathbb{R}^n$ is defined by the tuple $(G, c, A, b, \mathfrak{C})$ with $G \in \mathbb{R}^{n_c \times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c \times n_g}$, $b \in \mathbb{R}^{n_c}$, and $\mathfrak{C} := \{\mathcal{C}_1, \mathcal{C}_2 \ldots, \mathcal{C}_{n_p}\}$, where $\mathcal{C}_i \in \mathbb{R}^{m_i}$ are convex sets and $\sum_{i=1}^{n_p} m_i = n_g$, such that:

$$\mathcal{Z} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_{n_p}\}. \tag{1}$$

CCGs are a very general form of representing sets since ℓ_p norm balls, norm cones, among others can be represented directly. This entails that no approximation is required to represent ellipsoidal shapes, polytopes or even unbounded sets, which would otherwise introduce conservatism in the case of constrained zonotopes or polytopes. For instance, polytopes can be represented as

$$\mathcal{X} = (G, c, A, b, \|\xi\|_\infty \leq 1)\,, \tag{2}$$

and ellipsoids are defined as

$$\mathcal{X} = (G, c, [\,], [\,], \|\xi\|_2 \leq 1)\,. \tag{3}$$

Other types of sets can also be described as CCGs such as ellipsotopes, intervals, or zonotopes. For more information on CCGs, the reader is referred to [21]. The usual operations such as linear maps, Minkowski sum, and intersection are well-defined for CCGs and can be computed as in Definition 2.

Definition 2 ([21]): Consider three Constrained Convex Generators (CCGs) as in Definition 1:

- $\mathcal{Z} = (G_z, c_z, A_z, b_z, \mathfrak{C}_z) \subset \mathbb{R}^n$,
- $\mathcal{W} = (G_w, c_w, A_w, b_w, \mathfrak{C}_w) \subset \mathbb{R}^n$,
- $\mathcal{Y} = (G_y, c_y, A_y, b_y, \mathfrak{C}_y) \subset \mathbb{R}^m$,

and a matrix $R \in \mathbb{R}^{m \times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$\begin{aligned}
&R\mathcal{Z} + t = (RG_z, Rc_z + t, A_z, b_z, \mathfrak{C}_z),\\
&\mathcal{Z} \oplus \mathcal{W} =\\
&\left(\begin{bmatrix} G_z & G_w \end{bmatrix}, c_z + c_w, \begin{bmatrix} A_z & \mathbf{0} \\ \mathbf{0} & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_w\}\right),\\
&\mathcal{Z} \cap_R \mathcal{Y} =\\
&\left(\begin{bmatrix} G_z & \mathbf{0} \end{bmatrix}, c_z, \begin{bmatrix} A_z & \mathbf{0} \\ \mathbf{0} & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_y\}\right).
\end{aligned}$$

In this paper, we will only consider CCGs with generator sets defined by norm bounds. That is, we consider sets given by (1) where each generator set is

$$\mathcal{C}_i := \{\xi : \|\xi\|_{p_i} \leq 1\} \subset \mathbb{R}^{m_i}, \forall i \in 1, \cdots, n_p, \tag{4}$$

where

$$\sum_{i=1}^{n_p} m_i = n_g. \tag{5}$$

Note that when performing operations with two sets, the number of generators of the resulting set equals the sum of the generators of the original sets. Over numerous iterations, such as when computing the maximal invariant set, this leads to a significant increase in the number of generators, underscoring the necessity for an order reduction procedure.

III. Order Reduction Method of CCGs with Ellipsoids

The order of a CCG can be defined as $\frac{n_g}{n}$. It represents the computational burden required to describe the set. As mentioned earlier, in real applications, there will be limited computational power and a specified sampling time that must be respected. For instance, if we are computing whether the set will intersect with some obstacle, the solution to the optimization problem (irrespective of how it is formulated) must be found before the current iteration is over. Therefore, the maximum size of the data structures must be kept within a range such that the problem of interest can be tackled in the allotted time slot. As a result, it is sometimes essential to compute an overestimation to lower the complexity of the sets. This section describes this procedure for CCGs by proposing a method to eliminate generators, a method to eliminate constraints, and a guaranteed state estimation algorithm that uses the proposed methods.

A. Order Reduction

Before proceeding to the order reduction algorithm we need the following results. First, an ellipsoid can overapproximate a CCG with the following procedure, which amounts to enclose the overall generator set $\mathcal{C}_1 \times \ldots \times \mathcal{C}_{n_p}$ within an ellipsoid.

Lemma 1: Consider $\mathcal{Z} = \left(G, c, A, b, \{\mathcal{C}_1, \cdots \mathcal{C}_{n_p}\}\right)$, where G and A have the following structure

$$A := \begin{bmatrix} A_1 & A_2 & \ldots & A_{n_p} \end{bmatrix}, \tag{6a}$$
$$G := \begin{bmatrix} G_1 & G_2 & \ldots & G_{n_p} \end{bmatrix}, \tag{6b}$$

with $A_i \in \mathbb{R}^{n_c \times m_i}$ and $G_i \in \mathbb{R}^{n \times m_i}$, and the ellipsoid $\bar{\mathcal{Z}}$ given by

$$\bar{\mathcal{Z}} := \{\bar{G}\xi + c : \bar{A}\xi = b, \|\xi\|_2 \le 1\}, \tag{7}$$

with

$$\bar{A} := \begin{bmatrix} \frac{a_1}{\sqrt{d_1}} A_1 & \frac{a_2}{\sqrt{d_2}} A_2 & \ldots & \frac{a_{n_p}}{\sqrt{d_{n_p}}} A_{n_p} \end{bmatrix}, \tag{8a}$$
$$\bar{G} := \begin{bmatrix} \frac{a_1}{\sqrt{d_1}} G_1 & \frac{a_2}{\sqrt{d_2}} G_2 & \ldots & \frac{a_{n_p}}{\sqrt{d_{n_p}}} G_{n_p} \end{bmatrix}, \tag{8b}$$
$$a_i := \max\left(1, \frac{\sqrt{m_i}}{m_i^{\frac{1}{p_i}}}\right). \tag{8c}$$

If constants d_i, $i \in \{1, \ldots, n_p\}$ satisfy $d_i > 0$ and $\sum_{i=1}^{n_p} d_i = 1$, then, $\mathcal{Z} \subset \bar{\mathcal{Z}}$.

Proof: Note first that, as shown in [23], a_i is the lowest constant such that for all $\xi_i \in \mathbb{R}^{m_i}$,

$$\|\xi_i\|_2 \le a_i \|\xi_i\|_{p_i}. \tag{9}$$

Choose any $z \in \mathcal{Z}$, which is given by (1). Then, consider the following generator transformation

$$\tilde{\xi} := \begin{bmatrix} \frac{\sqrt{d_1}}{a_1}\xi_1 \\ \frac{\sqrt{d_2}}{a_2}\xi_2 \\ \vdots \\ \frac{\sqrt{d_{n_p}}}{a_{n_p}}\xi_{n_p} \end{bmatrix}. \tag{10}$$

From the definition of the 2-norm and (9) we obtain

$$\|\tilde{\xi}\|_2^2 = \sum_{i=1}^{n_p} \frac{d_i}{a_i^2} \|\xi_i\|_2^2 \le \sum_{i=1}^{n_p} d_i \|\xi_i\|_{p_i}^2. \tag{11}$$

Since the generator elements $\xi_i \in \mathbb{R}^{m_i}$ satisfy $\|\xi_i\|_{p_i} \le 1$ and given that $\sum_{i=1}^{n_p} d_i = 1$ we have that

$$\sum_{i=1}^{n_p} d_i \|\xi_i\|_{p_i}^2 \le \sum_{i=1}^{n_p} d_i = 1, \tag{12}$$

and therefore $\|\tilde{\xi}\|_2 \le 1$. If we consider $\tilde{\xi}$ as the new vector of generators, we have to rewrite the matrices as:

$$G\xi = \sum_{i=1}^{n_p} G_i \xi_i = \sum_{i=1}^{n_p} \bar{G}_i \frac{\sqrt{d_i}}{a_i} \xi_i = \bar{G}\tilde{\xi}, \tag{13a}$$
$$A\xi = \sum_{i=1}^{n_p} A_i \xi_i = \sum_{i=1}^{n_p} \bar{A}_i \frac{\sqrt{d_i}}{a_i} \xi_i = \bar{A}\tilde{\xi}, \tag{13b}$$

and we can conclude that $z \in \bar{\mathcal{Z}}$, given by (7) as we wanted to show. ∎

To determine the weights d_i one can set $d_i = \frac{1}{n_p}$ or, in the unconstrained case if more precision is required, solve the following convex optimization problem.

Lemma 2: Given the ellipsoidal overapproximation (7), for an unconstrained CCG choosing the weights d_i for $i \in \{1, \ldots, n_p\}$ that minimize the volume of $\bar{\mathcal{Z}}$ amounts to solving the following convex optimization problem.

$$\begin{aligned} \min_{d_i} \quad & -\log\left(\det\left(G^{\dagger\intercal} \operatorname{diag}\left(\frac{d_i}{a_i^2} I_{m_i}\right) G^{\dagger}\right)\right) \\ \text{s.t.} \quad & \sum_{i=1}^{n_p} d_i = 1. \end{aligned} \tag{14}$$

Proof: Given $x \in \mathbb{R}^n$ the vector ξ with minimum 2-norm such that

$$x = \bar{G}\xi \tag{15}$$

is given by

$$\xi = \bar{G}^{\dagger} x. \tag{16}$$

Note that since

$$\bar{G} = G \operatorname{diag}\left(\frac{a_i}{\sqrt{d_i}} I_{m_i}\right), \tag{17}$$

where in $\operatorname{diag}\left(\frac{a_i}{\sqrt{d_i}} I_{m_i}\right)$, $i \in \{1, \ldots, n_p\}$, we have that

$$\bar{G}^{\dagger} = \operatorname{diag}\left(\frac{\sqrt{d_i}}{a_i} I_{m_i}\right) G^{\dagger}. \tag{18}$$

Therefore $\|\xi\|_2 \le 1$ is equivalent to

$$x^\intercal G^{\dagger\intercal} \operatorname{diag}\left(\frac{d_i}{a_i^2} I_{m_i}\right) G^{\dagger} x \le 1. \tag{19}$$

The result ensues by noting that minimizing the volume of an ellipsoid given by (19) is equivalent to maximizing

$$\det\left(G^{\dagger\intercal} \operatorname{diag}\left(\frac{d_i}{a_i^2} I_{m_i}\right) G^{\dagger}\right), \tag{20}$$

that $\log(\cdot)$ is a monotonically increasing function and that $-\log(\det(\cdot))$ is a convex function. ∎

Based on Lemmas 11 and 12 in [15], a constrained ellipsoid can always be expressed without constraints as in the following Lemma, which corresponds to Lemma 13 in [15], making explicit the expressions for matrices T and t.

Lemma 3: Given a constrained ellipsoid of the form

$$\mathcal{X} = (G, c, A, b, \|\xi\|_2 \le 1), \tag{21}$$

it can be expressed with $n_g - n_c$ generators as

$$\mathcal{X} \equiv \tilde{\mathcal{X}} := (GT, c + Gt, [\,], [\,], \|\xi\|_2 \le 1), \tag{22}$$

where

$$t := A^{\dagger} b, \tag{23a}$$
$$T := \sqrt{1 - \|t\|^2}\, \operatorname{null}(A), \tag{23b}$$

$A^{\dagger}$ is the pseudo-inverse of A and $\operatorname{null}(A)$ is an orthonormal basis of the null-space of A.

Proof: To prove the lemma we must show that these sets are equivalent

$$\{\xi : A\xi = b, \|\xi\|_2 \le 1\} \equiv \left\{T\tilde{\xi} + t : \|\tilde{\xi}\|_2 \le 1\right\}. \tag{24}$$

This can be shown as follows. In the first set, we can perform the transformation $\xi = \bar{\xi} + t$, and noting that, from (23a), $At = b$ we obtain that

$$\{\xi : A\xi = b, \|\xi\|_2 \le 1\} \tag{25}$$

is equivalent to

$$\left\{\bar{\xi} + t : A\bar{\xi} = \mathbf{0}, \|\bar{\xi} + t\|_2 \le 1\right\}. \tag{26}$$

Given that $A\bar{\xi} = \mathbf{0}$, $\bar{\xi}$ is in the nullspace of A, and therefore without loss of generality we can make the transformation $\bar{\xi} = T\tilde{\xi}$. Given that, from (23), $A^{\dagger\intercal}\operatorname{null}(A) = \mathbf{0}$ and $t^\intercal T = \mathbf{0}$, we have,

$$\|T\tilde{\xi} + t\|_2 = \sqrt{\|T\tilde{\xi}\|_2^2 + \|t\|_2^2}. \tag{27}$$

Then, from the fact that $\|T\tilde{\xi}\|_2 = \sqrt{1 - \|t\|_2^2}\|\tilde{\xi}\|_2$ we have that (26) is equivalent to

$$\left\{T\tilde{\xi} + t : \|\tilde{\xi}\|_2 \le 1\right\}, \tag{28}$$

as we wanted to show. ■

Finally, an ellipsoid can always be expressed with a vector of generators of size $\operatorname{rank}(G)$ with the following method

Lemma 4: An ellipsoid of the form

$$\mathcal{X} = (G, c, [\,], [\,], \|\xi\|_2 \le 1), \tag{29}$$

can always be expressed with $\operatorname{rank}(G)$ generators by taking the singular value decomposition of G

$$USV^\intercal = G \tag{30}$$

and computing

$$\mathcal{X} \equiv \tilde{\mathcal{X}} := (US, c, [\,], [\,], \|\xi\|_2 \le 1). \tag{31}$$

Proof: Performing the generator transformation $\tilde{\xi} = V^\intercal\xi$, since V is orthonormal we have that

$$\|\tilde{\xi}\|_2 \le \|\xi\|_2 \le 1, \tag{32}$$

thus concluding the proof. ■

A similar exact order reduction method for ellipsoids was presented in [15]. However, since the order reduction method of Lemma 4 consists of performing a singular value decomposition on $G \in \mathbb{R}^{n\times n_g}$ its complexity is $\mathcal{O}(nn_g^2)$, whereas the complexity of the method in [15] is $\mathcal{O}(n^3 + n_g^3)$.

Since we are interested in reducing the number of generators without changing the effect of some or most of the generators, we adopt a method based on the lift-then-reduce strategy for CZs [14] to partially reduce the order of a CCG.

Lemma 5: Consider a CCG of the form (1) partition G and A as

$$A := \begin{bmatrix}\bar{A}_1 & \bar{A}_2\end{bmatrix}, \tag{33a}$$
$$G := \begin{bmatrix}\bar{G}_1 & \bar{G}_2\end{bmatrix}, \tag{33b}$$

where

$$\bar{A}_1 := \begin{bmatrix}A_1 & \dots & A_{\bar{n}_p}\end{bmatrix}, \tag{34a}$$
$$\bar{A}_2 := \begin{bmatrix}A_{\bar{n}_p+1} & \dots & A_{n_p}\end{bmatrix}, \tag{34b}$$
$$\bar{G}_1 := \begin{bmatrix}G_1 & \dots & G_{\bar{n}_p}\end{bmatrix}, \tag{34c}$$
$$\bar{G}_2 := \begin{bmatrix}G_{\bar{n}_p+1} & \dots & G_{n_p}\end{bmatrix}, \tag{34d}$$

for some integer $\bar{n}_p$ such that $1 \le \bar{n}_p \le n_p$ Given matrices $\tilde{G}_2 \in \mathbb{R}^{n\times(n+n_c)}$ and $\tilde{A}_2 \in \mathbb{R}^{n_c\times(n+n_c)}$ that yield the following ellipsoidal overbound

$$\begin{aligned}&\left(\begin{bmatrix}\bar{G}_2\\ \bar{A}_2\end{bmatrix}, \mathbf{0}, [\,], [\,], \mathcal{C}_{\bar{n}_p+1} \times \ldots \times \mathcal{C}_{n_p}\right) \subset\\ &\subset \left(\begin{bmatrix}\tilde{G}_2\\ \tilde{A}_2\end{bmatrix}, \mathbf{0}, [\,], [\,], \|\xi\|_2 \le 1\right),\end{aligned} \tag{35}$$

the CCG (1) can be overapproximated by

$$\left(\begin{bmatrix}\bar{G}_1 & \tilde{G}_2\end{bmatrix}, c, \begin{bmatrix}\bar{A}_1 & \tilde{A}_2\end{bmatrix}, b, \mathcal{C}_1 \times \ldots \times \mathcal{C}_{\bar{n}_p} \times \tilde{\mathcal{C}}\right), \tag{36}$$

where

$$\tilde{\mathcal{C}} := \{\xi : \|\xi\|_2 \le 1\} \subset \mathbb{R}^{n+n_c}. \tag{37}$$

Proof: Choose any $z \in \mathcal{Z}$. First note that for (1) stating that $z \in \mathcal{Z}$ is the same as stating that

$$\begin{bmatrix}z\\ \mathbf{0}\end{bmatrix} \in \bar{\mathcal{Z}} := \left(\begin{bmatrix}G\\ A\end{bmatrix}, \begin{bmatrix}c\\ -b\end{bmatrix}, [\,], [\,], \mathcal{C}_1 \times \ldots \times \mathcal{C}_{n_p}\right). \tag{38}$$

Notice that $\bar{\mathcal{Z}}$ can be expressed as $\bar{\mathcal{Z}} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ where

$$\mathcal{Z}_1 := \left(\begin{bmatrix}\bar{G}_1\\ \bar{A}_1\end{bmatrix}, \begin{bmatrix}c\\ -b\end{bmatrix}, [\,], [\,], \mathcal{C}_1 \times \ldots \times \mathcal{C}_{\bar{n}_p}\right), \tag{39a}$$
$$\mathcal{Z}_2 := \left(\begin{bmatrix}\bar{G}_2\\ \bar{A}_2\end{bmatrix}, \mathbf{0}, [\,], [\,], \mathcal{C}_{\bar{n}_p+1} \times \ldots \times \mathcal{C}_{n_p}\right). \tag{39b}$$

From (35) we have that $\mathcal{Z}_2 \subset \bar{\bar{\mathcal{Z}}}_2$ where

$$\bar{\bar{\mathcal{Z}}}_2 := \left(\begin{bmatrix}\tilde{G}_2\\ \tilde{A}_2\end{bmatrix}, \mathbf{0}, [\,], [\,], \|\xi\|_2 \le 1\right). \tag{40}$$

Using the expression for the Minkowski sum in Definition 2 we have that $\begin{bmatrix}z^\intercal & \mathbf{0}^\intercal\end{bmatrix}^\intercal$ is in the set

$$\left(\begin{bmatrix}\bar{G}_1 & \tilde{G}_2\\ \bar{A}_1 & \tilde{A}_2\end{bmatrix}, \begin{bmatrix}c\\ -b\end{bmatrix}, [\,], [\,], \mathcal{C}_1 \times \ldots \times \mathcal{C}_{\bar{n}_p} \times \tilde{\mathcal{C}}\right), \tag{41}$$

and we can conclude that z is in the set given by (36) as we wanted to show. ■

Since we aim to eliminate only some of the generators we require a heuristic to select which generators to eliminate. Given Lemmas 1 and 3, one possible heuristic to estimate the size of the contribution of each generator to the final set is the following

Definition 3: Partition the matrix T from (23) as

$$T := \begin{bmatrix}T_1\\ T_2\\ \vdots\\ T_{n_p}\end{bmatrix}, \tag{42}$$

where $T_i \in \mathbb{R}^{m_i\times(n_g-n_c)}, \forall i \in \{1, \cdots, n_p\}$. Then, the weight of each generator is defined as

$$w_i := a_i\|G_iT_i\|_2, \forall i \in \{1, \cdots, n_p\}, \tag{43}$$

where a_i is defined in (8c).

The transformation T_i maps the new generators, constrained only by the two-norm, back to the original generator space, while G_i transforms from the space of the original generators to the space of the set. Consequently, w_i is an upper bound on the radius of a spheroid enclosing the generator set $\mathcal{C}_i$, which serves as an approximation of the volume contribution attributed to individual generator sets. This approximation can be applied to any type of p-norm.

Given these methods, the order reduction algorithm to achieve a desired order r is given by Algorithm 1. We note that in practice, as will be seen in Section IV, it is often preferable to set $d_i = \frac{1}{n_p}$ than to solve (14) due to the increase in computational time that it involves. In this situation, the computational complexity of the algorithm is $\mathcal{O}(n_c n_g^2 + n(n_g - rn)^2)$.

Algorithm 1 Order reduction algorithm $\tilde{\mathcal{Z}} = \mathrm{ord\_red}(\mathcal{Z}, r)$.

Require: $\mathcal{Z}$, r

Reorder the generators such that $w_{i+1} \leq w_i$.
Set $\bar{n}_p$ as the lowest number with $rn \leq \sum_{i=1}^{\bar{n}_p} m_i$.
Partition A and G as in (33a) and (33b)
Compute the overapproximation (35) using Lemma 1 either by setting $d_i = \frac{1}{n_p}$ or by solving (14), and Lemma 4.
Compute (36)

Remark 1: The adoption of the 2-norm is motivated by its compatibility with Lemma 3 and Definition 3, enabling the derivation of a heuristic for estimating the size of the reduced generator set. It is worth noting that our approach is flexible, as demonstrated by the potential adaptation of Lemma 1 to accommodate a variety of p-norms, thereby extending the applicability of the method to different norm settings.

B. Constraint Reduction

As in [14] to remove the constraint i and the generator j from the CCG we consider the following overapproximation, which adapts Proposition 5 in [14] to CCGs:

Lemma 6: A CCG of the form (1) satisfies

$$\begin{aligned}&(G, c, A, b, \mathfrak{C}) \subset \\ &\subset (G - \Lambda_G A, c + \Lambda_G b, A - \Lambda_A A, b - \Lambda_A b, \mathfrak{C}) \end{aligned} \tag{44}$$

for every $\Lambda_G \in \mathbb{R}^{n \times n_c}$ and $\Lambda_A \in \mathbb{R}^{n_c \times n_c}$.

Proof: $z \in (G, c, A, b, \mathfrak{C})$ if there exists $\xi \in \mathfrak{C}$ such that

$$\begin{bmatrix} z \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} G \\ A \end{bmatrix} \xi + \begin{bmatrix} c \\ -b \end{bmatrix}. \tag{45}$$

For any such ξ

$$\begin{bmatrix} z \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} G \\ A \end{bmatrix} \xi + \begin{bmatrix} c \\ -b \end{bmatrix} + \begin{bmatrix} \Lambda_G \\ \Lambda_A \end{bmatrix} (b - A\xi). \tag{46}$$

Therefore, $z \in (G - \Lambda_G A, c + \Lambda_G b, A - \Lambda_A A, b - \Lambda_A b, \mathfrak{C})$. ■

In this paper, instead of considering the Hausdorff distance to select which generator to remove as in [14], for each constraint i we consider removing the generator $j = \arg\max_k |a_{ik}|$. To remove one constraint we select the one which yields the smaller 2-norm of the overapproximation when removed. To remove the constraint i and the generator j we select, from [14],

$$\Lambda_G := G E_{ji} a_{ij}^{-1}, \tag{47}$$

$$\Lambda_A := A E_{ji} a_{ij}^{-1}, \tag{48}$$

where a_{ij} is the element of A in the ith row and jth column and $E_{ji} \in \mathbb{R}^{n_g \times n_c}$ is zero except for a one in the (j, i) position. Note that the ith row and jth column of $A - \Lambda_A A$, the jth row of $G - \Lambda_G A$ and the ith element of $b - \Lambda_A b$ are zero. Therefore, constraint i and generator element j may be removed from the set by removing these rows and columns and considering a generator set $\tilde{\mathfrak{C}}$ which is the projection of $\mathfrak{C}$ in the hyperplane given by fixing the dimension j. That is, suppose that the generator j corresponds to the generator set l, that is,

$$\sum_{i=1}^{l-1} m_i < j \leq \sum_{i=1}^{l} m_i, \tag{49}$$

then, if $m_l = 1$, computing $\tilde{\mathfrak{C}}$ amounts to remove $\mathcal{C}_l$ from $\mathfrak{C}$, that is

$$\tilde{\mathfrak{C}} := \mathcal{C}_1 \times \ldots \times \mathcal{C}_{l-1} \times \mathcal{C}_{l+1} \times \ldots \times \mathcal{C}_{n_p}. \tag{50}$$

If $m_l > 1$ then we must replace $\mathcal{C}_l$ by $\tilde{\mathcal{C}}_l$ where

$$\tilde{\mathcal{C}}_l := \{\xi : \|\xi\|_{p_l} \leq 1\} \subset \mathbb{R}^{m_l - 1} \tag{51}$$

that is,

$$\tilde{\mathfrak{C}} := \mathcal{C}_1 \times \ldots \times \tilde{\mathcal{C}}_l \times \ldots \times \mathcal{C}_{n_p}. \tag{52}$$

In summary, the constraint reduction algorithm is given by Algorithm 2, which has a complexity of $\mathcal{O}((n_c - \tilde{n}_c) n_c^2 n_g^2)$.

Algorithm 2 Constraint reduction algorithm $\tilde{\mathcal{Z}} = \mathrm{con\_red}(\mathcal{Z}, \bar{n}_c)$.

Require: $\mathcal{Z}$, $\bar{n}_c$

for $k \leftarrow 1$ to $n_c - \bar{n}_c$ **do**
  Set $\mathcal{Z}^{prev} = \mathcal{Z} \equiv (G, c, A, b, \mathfrak{C})$
  Set $weight = \infty$
  for $i \leftarrow 1$ to $n_c - k + 1$ **do**
    Set $j = \arg\max_l |a_{il}|$
    Compute $\mathcal{Z}^{prev} \subset \left(\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}, \tilde{\mathfrak{C}}\right)$ as follows:
    Compute $\Lambda_G := G E_{ji} a_{ij}^{-1}$.
    Compute $\Lambda_A := A E_{ji} a_{ij}^{-1}$.
    $\tilde{G}$ is $G - \Lambda_G A$ removing column j.
    $\tilde{c} = c - \Lambda_G b$
    $\tilde{A}$ is $A - \Lambda_A A$ removing column j and row i.
    $\tilde{b}$ is $b - \Lambda_A b$ removing row i.
    $\tilde{\mathfrak{C}}$ is (50) or (52).
    Using Lemmas 1 and 3 compute:
    $\left(\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}, \tilde{\mathfrak{C}}\right) \subset \left(\bar{G}, \bar{c}, [\,], [\,], \|\xi\|_2 \leq 1\right)$
    Compute *new_weight* $= \|\bar{G}\|_2$
    if *new_weight* $<$ *weight* **then**
      Set $\mathcal{Z} = \left(\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}, \tilde{\mathfrak{C}}\right)$
      Set *weight* $=$ *new_weight*
    end if
  end for
end for

C. Guaranteed State Estimation

The problem of guaranteed state estimation in discrete-time LTI systems can be formulated as the problem of finding a set of possible state values given measurements, disturbance, noise, and initial state bounds. The model is provided by:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{53a}$$
$$y_k = Cx_k + v_k, \tag{53b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathcal{W} \subset \mathbb{R}^n$, $y_k \in \mathbb{R}^{n_y}$, and $v_k \in \mathcal{V} \subset \mathbb{R}^{n_y}$ represent the system state, input, disturbance signal, output, and noise, respectively. Given the operations defined in Definition 2, one may estimate the state recursively, since given a set $X_k \subset \mathbb{R}^n$ such that $x_k \in \mathcal{X}_k$ and a measurement y_k, the set $\mathcal{X}_{k+1} \subset \mathbb{R}^n$ such that $x_{k+1} \in \mathcal{X}_{k+1}$ can be computed as

$$\mathcal{X}_{k+1} = (A\mathcal{X}_k \oplus \mathcal{W} + Bu_k) \cap_C (y_k - \mathcal{V}). \tag{54}$$

In this implementation, we assume that the sets $\mathcal{W}$ and $\mathcal{V}$ are represented as CCGs. In particular, we consider that

$$\mathcal{W} := (G_w, c_w, [\,], [\,], \mathfrak{C}_w), \tag{55a}$$
$$\mathcal{V} := (G_v, c_v, [\,], [\,], \mathfrak{C}_v), \tag{55b}$$

with $G_w \in \mathbb{R}^{n \times n_w}$ and $G_v \in \mathbb{R}^{n_y \times n_v}$.

In order to maintain the complexity of the set description manageable and to keep the computational burden low we adopt the following event-triggering order reduction mechanism, where $0 < \beta < 1$ must be small enough.

Algorithm 3 Set-valued CCG observer with event triggered order reduction

Require: $\mathcal{X}_0$, $\mathcal{V}$, $\mathcal{W}$, T_c, β
Set $\mathcal{X}_0^r = X_0$
for $k \geq 0$ **do**
  $\mathcal{X}_{k+1} = (A\mathcal{X}_k \oplus \mathcal{W} + Bu_k) \cap_C (y_k - \mathcal{V})$
  $\mathcal{X}_{k+1}^r = (A\mathcal{X}_k^r \oplus \mathcal{W} + Bu_k) \cap_C (y_k - \mathcal{V})$
  Try to compute under a time of T_c:
  $\mathcal{X}_{k+1}^{or} = \text{con\_red}(\mathcal{X}_{k+1}, \beta n_c)$.
  $\mathcal{X}_{k+1}^{or} = \text{ord\_red}\left(\mathcal{X}_{k+1}^{or}, \beta \frac{n_g}{n}\right)$.
  if Computation was successful **then**
    $\mathcal{X}_{k+1}^r = \mathcal{X}_{k+1}^{or}$
  else
    $\mathcal{X}_{k+1} = \mathcal{X}_{k+1}^r$
  end if
end for

The approach in Algorithm 3 guarantees that the computation time at each step does not exceed T_c but may be too conservative. A less conservative solution that does not guarantee a fixed upper bound on the computation time but ensures that the computation time is approximately T_d is given by Algorithm 4

Algorithm 4 Set-valued CCG observer with event triggered order reduction

Require: $\mathcal{X}_0$, $\mathcal{V}$, $\mathcal{W}$, T_d
Set $\mathcal{X}_0^r = \mathcal{X}_0$
for $k \geq 0$ **do**
  start the clock.
  $\mathcal{X}_{k+1} = (A\mathcal{X}_k \oplus \mathcal{W} + Bu_k) \cap_C (y_k - \mathcal{V})$
  $\mathcal{X}_{k+1}^{or} = \text{con\_red}(\mathcal{X}_{k+1}, n_c - 2n_y)$.
  $\mathcal{X}_{k+1}^{or} = \text{ord\_red}\left(\mathcal{X}_{k+1}^{or}, \frac{n_g - 2(n_w + n_v) - (n_c - 2n_y)}{n}\right)$.
  if Elapsed time is greater than T_d **then**
    $\mathcal{X}_{k+1} = \mathcal{X}_{k+1}^{or}$
  end if
end for

IV. Numerical Results

In order to assess the performance of the proposed methods, we start by considering the order reduction applied to random constrained zonotopes in $\mathbb{R}^{10}$ on an Intel Core i7-12700H processor at 2.70 GHz. We consider that each element of G is drawn from a normal distribution centred at zero with a standard deviation of $\frac{1}{n_g}$, where n_g is the number of generators, each element of A is drawn from a normal distribution centred at zero with a standard deviation of one, and each element of b is drawn from a uniform distribution from -0.5 to 0.5. The order of a set representation is given by $\frac{n_g}{n}$. We consider an order reduction of the form

$$\tilde{\mathcal{Z}} = \text{ord\_red}(\mathcal{Z}, \frac{\tilde{n}_g}{n}). \tag{56}$$

Finally, we overbound the result with a CZ by changing the generator vector noting that

$$\{\xi : \|\xi\|_2 \leq 1\} \subset \{\xi : \|\xi\|_\infty \leq 1\} \tag{57}$$

and compare the results obtained with the order reduction method proposed in this paper with that of [14] for CZs and that of [15]. We represent a CZ as a CCG with $n_p = n_g$. We note that, in practice, the required time to solve (14) is much larger than the remaining of the algorithm and yields only marginal improvements. Therefore, we will consider $d_i = \frac{1}{n_p}$.

To have a more systematic assessment of the performance of the proposed method in Figures 1 and 2 we present the results of the difference of the obtained volume overapproximation of the enclosing hyperrectangles with both methods, $\frac{V}{V_{original}}$ where V is the volume of the enclosing hyperrectangle of the reduced set and $V_{original}$ is the volume of the enclosing hyperrectangle of the original set, and the computational time, respectively for different randomly generated CZs with different number of generators n_g while setting the number of constraints to $n_c = 0.4n_g$.

We can observe from Figure 1 that in this case, for low order proportions, [14] and [15] fare worse than the method of Algorithm 1 in terms of volume of the resulting set, while for higher order proportions, the volume of the resulting set with Algorithm 1 is higher, which may be partly due to the near-ellipsoidal shape of the described set. This trend shows that Algorithm 1 is competitive in situations that require sets expressed with a small order. From Figure 2 we observe that in terms of computational time, there is no clear advantage to either method.

To assess the performance of the constraint reduction method, we consider randomly generated CZs as before. We

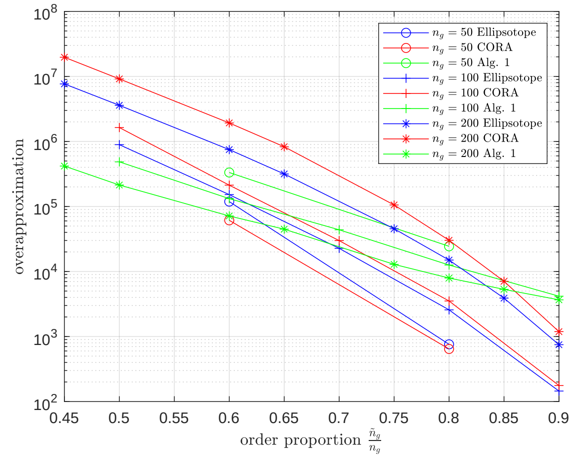

Fig. 1. Average of the obtained volume overapproximation $\frac{V}{V_{original}}$ with the proposed order reduction method (Alg. 1), the order reduction method in [14] (CORA), and the order reduction method in [15] for ellipsotopes (Ellipsotope), for different randomly generated CZs with different numbers of generators n_g. This would correspond to a uniform expansion on each side of $(V/V_{original})^{\frac{1}{10}}$ times.

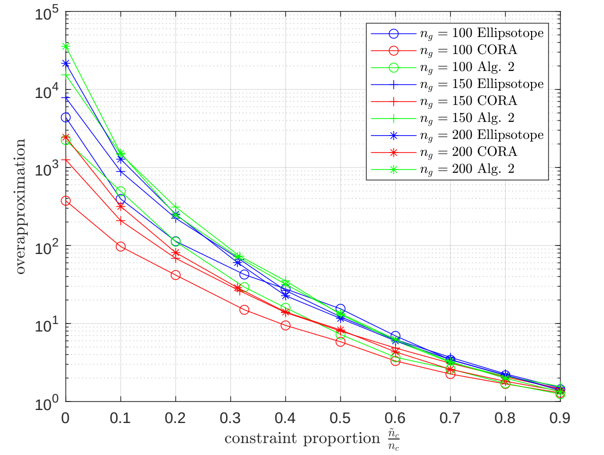

Fig. 3. Average of the obtained volume overapproximation $\frac{V}{V_{original}}$ with the proposed constraint reduction method (Alg. 2), the constraint reduction method in [14] (CORA), and the constraint reduction method in [15] for ellipsotopes (Ellipsotope), for different randomly generated CZs with different numbers of generators n_g.

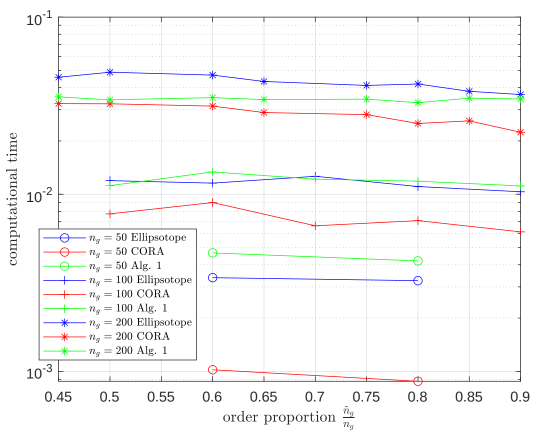

Fig. 2. Average of the computational times with the proposed order reduction method (Alg. 1), the order reduction method in [14] (CORA), and the order reduction method in [15] for ellipsotopes (Ellipsotope), for different randomly generated CZs with different numbers of generators n_g.

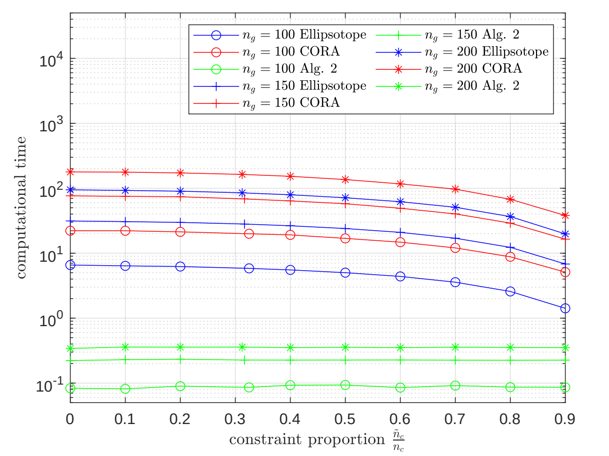

Fig. 4. Average of the computational times with the proposed constraint reduction method (Alg.2) and the constraint reduction method in [14] (CORA), and the constraint reduction method in [15] for ellipsotopes (Ellipsotope), for different randomly generated CZs with different numbers of generators n_g.

consider a constraint reduction of the form

$$\tilde{\mathcal{Z}} = \text{con\_red}(\mathcal{Z}, \tilde{n}_c). \tag{58}$$

Again, for a systematic assessment of the performance of the proposed constraint reduction method in Figures 3 and 4 we present the results of the difference of the obtained volume overapproximation and the computational times, respectively for different randomly generated CZs with different number of generators n_g, while setting the number of constraints to $n_c = 0.4n_g$.

We can observe from Figures 3 and 4 that in this case there is no clear advantage of either method regarding the volume of the overapproximation. Still, there are significant savings regarding computational time for Algorithm 2.

Regarding guaranteed state estimation, we test Algorithm 3 on system (53) with no input and $A \in \mathbb{R}^{3\times3}$ and $C \in \mathbb{R}^{3\times3}$ are random orthonormal matrices. The initial state, the process disturbance and the measurement noise satisfy $\|x_0\|_\infty \leq 100$, $\|w_k\|_\infty \leq 1$ and $\|v_k\|_\infty \leq 1$ for all k.

Figure 5 shows the mean over time of the volume of the state estimate for the method for CCGs proposed here with event-triggered order reduction (Algorithm 3), with $T_c = 0.5$ and $\beta = 0.6$ and the same algorithm using a CZ approximation with the order reduction method in [14] for CZs, with $\beta = 0.1$. Figure 6 shows the time to compute the description of the set at runtime. We can observe that with the methods proposed here, we obtain better overapproximations because the number of generators used to describe the set is larger.

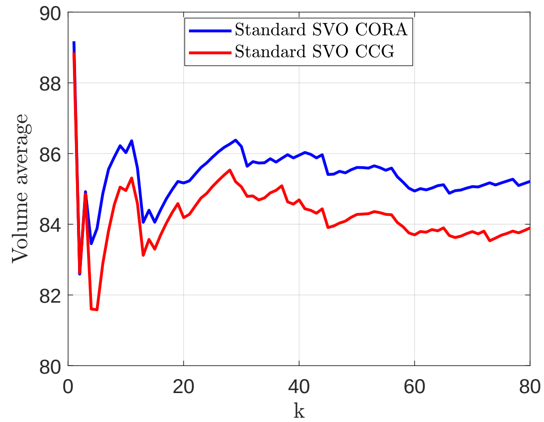

Fig. 5. Volume mean $\sum_{j=1}^{k} \frac{Vol_j}{k}$, where Vol_k is the volume at time k, of the state estimate at various iterations for the two set-valued observers (SVO).

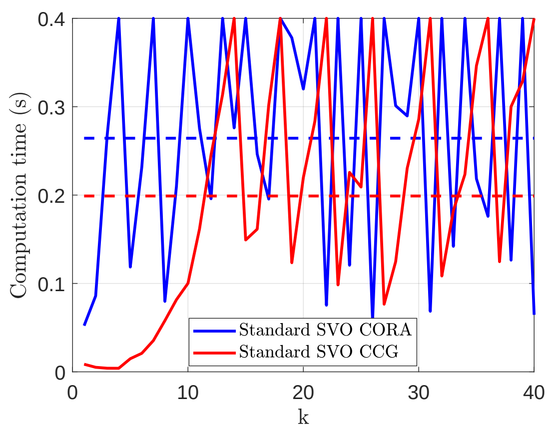

Fig. 6. Computation times for the two set-valued observers (SVO). The dashed horizontal lines correspond to the mean value for each method.

V. Conclusions

We proposed an ellipsoid-based method for order reduction of CCGs, which is shown to perform similarly to, and in some cases better than existing approaches such as conventional CZ order reduction methods. Through numerical examples, we also demonstrated its benefits in the context of guaranteed state estimation. Overall, our proposed ellipsoid-based technique for CCG order reduction provides a promising approach for efficiently and accurately approximating convex sets, which has important implications for various applications in control and robotics. Future developments may include the development of order reduction methods for more general generator sets than (4) and the application of these methods to nonlinear reachability analysis.

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